Abstract—This paper studies existence of Zeno behavior in electromechanical hybrid systems, giving conditions for the existence of Zeno behavior and verifying these conditions through experimentation. To achieve these results, we begin by considering a special class of hybrid electromechanical systems modeling mechanical systems undergoing impacts and driven by DC motors. Formal conditions for the existence of Zeno behavior in systems of this form are given, and these conditions are used to complete the hybrid system model to allow for solutions to be carried past the Zeno point—this results in periods of unconstrained and constrained motion, with transitions to the constrained motion occurring at the Zeno point. Utilizing this formal theory, we consider a double pendulum system with a mechanical stop where the top link is driven by a permanent magnet DC motor and demonstrate that, due to the mechanical stop, the electromechanical hybrid model for this system displays Zeno behavior. As a result, this model is completed and, through simulation, we find a periodic orbit in this completed system, i.e., a Zeno periodic orbit. We then verify, through experimentation, that the formal methods developed in this paper provide an accurate description of the behavior of the physical system—the Zeno periodic orbit found through simulation occurs on the physical system.

I. INTRODUCTION

Hybrid dynamical systems are systems that display both continuous and discrete behavior \[9\], \[16\] and \[27\]. As such, they describe a large class of physical systems, especially those undergoing impacts. A fundamental phenomenon which is unique to hybrid systems is Zeno behavior, where an infinite number of discrete transitions occur in a finite amount of time. Since its introduction, Zeno behavior has been well studied due to its uniqueness to hybrid systems, and the way in which it prevents the extension of standard notions of solutions to the hybrid framework; Zeno solutions, by definition, only exist for a finite period of time.

Zeno behavior has been well studied in the hybrid systems community for over 10 years now \[2\], \[7\], \[8\], \[10\], \[11\], \[24\], \[29\], with the first paper that the authors are aware of discussing the phenomena in the context of hybrid systems being \[12\]. Even though a lot of ground has been covered with respect to the analysis of Zeno behavior, the hybrid systems community still remains divided over its existence in the real world. One side of the discussion claims that since it occurs as a result of instantaneous discrete changes in a system, which cannot occur in reality, Zeno behavior itself does not occur in reality. As a result, Zeno behavior is thus not interesting, and instead the model of the system being studied should be refined so that Zeno behavior does not occur since it does not occur in reality. The other side of the discussion claims that although Zeno behavior does not occur in reality, modeling of systems with instantaneous discrete changes is “close” to reality, and therefore system models with Zeno behavior will display behavior that is “close” to the physical behavior—it is therefore important to study Zeno behavior. The authors, admittedly, come from the latter camp and have established numerous formal results related to Zeno behavior. Specifically, results that relate Zeno behavior to a type of equilibria unique to hybrid systems, termed Zeno equilibria, and the existence of Zeno behavior to the stability of these equilibria (see \[21\], \[15\], \[13\]). This allowed for conditions for the existence of Zeno behavior, and for hybrid models to be completed so that Zeno solutions can be extended beyond their finite limit points.

This paper will present a physical grounding for the formal ideas that have been considered relating to Zeno behavior, Zeno equilibria, and completing hybrid system models to allow for the simulation of Zeno hybrid systems. In particular, the primary objective of this paper is to apply these formal ideas to model of a nontrivial mechanical system undergoing impacts, i.e., a hybrid mechanical system. It is important to note that there are many “simpler” models of hybrid mechanical systems that can be considered, such as the infamous bouncing ball, but the goal of this paper is to consider a system with nonlinear dynamics, and nontrivial impact behaviors. At the same time, realizing impacts in a purely mechanical system does not encompass real world behavior, since mechanical systems are generally connected to actuators which are electrical. Therefore, considering an electromechanical system with nonlinear dynamics and nontrivial impact behaviors provides a true testbed for the theory that will be applied.

The specific electromechanical hybrid system that this paper considers is a double pendulum with a mechanical stop. This system consists of two links, for which the top link is actuated by a Permanent Magnet DC motor through P-D control, and the bottom link is constrained in its motion by a mechanical stop; this stop results in impacts in the system, making it a hybrid system. The reason for choosing this system as a testbed for
formal results in hybrid systems is because, again, it is a non-trivial hybrid system. More generally, and more importantly, this system is meant to represent a much more interesting class of hybrid systems: bipedal robots. In particular, our double pendulum models the leg of a bipedal robot with the mechanical stop playing the role of a “knee-cap” (see [17]).

The first half of this paper is devoted to developing the formal tools necessary to understand Zeno behavior in a class of elecromechanical hybrid systems. In particular, in Section II will start with reviewing Lagrangian hybrid systems and extend these definitions to a special class of electromechanical systems: those modeling DC motors. The end result is a formal hybrid model of systems undergoing impacts and driven by DC motors. In Section III, Zeno behavior is introduced and sufficient conditions for the existence of Zeno behavior in the special class of electromechanical hybrid systems considered in this paper are presented. After detecting Zeno behavior, it will be necessary to complete this system to allow solutions to be carried past the Zeno point; the details of this completion construction will be discussed in Section III.

The second half of this paper is devoted to verifying the formal results of this paper through experimentation, i.e., verifying the existence of Zeno behavior on the double pendulum powered by a DC motor both through simulation and experimental validation. In Section V we introduce a model of the double pendulum, detect Zeno behavior in this model, and compute this system to allow for solutions to be carried past the Zeno points. After performing this process, we are able to find a Zeno periodic orbit in simulation. Section VI considers the real double pendulum and relates its behavior to the behavior discovered in simulation. In particular, just as with the simulated system, we find that the physical system has a Zeno periodic orbit. Moreover, the Zeno behavior in simulation correctly predicts the impact behavior of the physical system. We, therefore, have provided evidence that Zeno behavior provides a good modeling paradigm for the behavior of real physical systems.

II. ELECTROMECHANICAL HYBRID SYSTEMS

In this section, we introduce the notion of a Lagrangian system, the elecromechanical system, the resulting Extended Lagrangian system and eventually the associated Extended Lagrangian hybrid system. This section will also discuss the presence of holonomic and unilateral constraints that will be important due to the mechanical stop. Hybrid systems of this form have been studied in the context of Lagrangian hybrid systems in Zeno behavior, see [2], [4], [14], and were also formulated as linear complementarity systems in [18] and [25]. Extended Lagrangian systems were discussed in the area of electromechanical systems in [28].

A. Hybrid Systems:

Definition 1: We will start by reviewing the definition of a simple hybrid system (A graphical representation can be seen in Fig. 1). A simple hybrid system is a tuple

\[ \mathcal{H} = (D, G, R, f), \]

where

- \( D \) is a smooth manifold called the domain,
- \( G \) is an embedded submanifold of \( D \) called the guard,
- \( R \) is a smooth map \( R : G \to D \) called the reset map,
- \( f \) is a smooth vector field on the manifold \( D \).

This paper focuses on simple hybrid systems, having a single domain, guard and reset map. A general hybrid system (see [3]), which is not discussed here, consists of a collection of domains, guards, reset maps and vector fields as indexed by an oriented graph.

Definition 2: An execution of a simple hybrid system \( \mathcal{H} \) is a tuple \( \chi = (\Lambda, \mathcal{I}, \mathcal{C}) \), where

- \( \Lambda = \{0, 1, 2, \ldots\} \subseteq \mathbb{N} \) is an indexing set,
- \( \mathcal{I} = \{I_i\}_{i \in \Lambda} \) where for each \( i \in \Lambda, I_i \) is defined as follows: \( I_i = [t_i, t_{i+1}] \) if \( i, i + 1 \in \Lambda \) and \( I_{N-1} = [t_{N-1}, t_N] \) or \( [t_{N-1}, t_N) \) or \( [t_{N-1}, \infty) \) if \( |\Lambda| = N \), \( N \) finite. Here, for all \( i, i + 1 \in \Lambda \), \( t_i \leq t_{i+1} \) with \( t_i, t_{i+1} \in \mathbb{R} \), and \( t_{N-1} \leq t_N \) with \( t_{N-1}, t_N \in \mathbb{R} \),
- \( \mathcal{C} = \{G_i\}_{i \in \Lambda} \) is a set of continuous trajectories, and they must satisfy \( c_i(t) = f_{\rho(i)}(c_{i+1}(t)) \) for \( t \in I_i \).

We require that when \( i, i + 1 \in \Lambda \),

1. \( c_i(t_{i+1}) \in G_i \),
2. \( R(c_i(t_{i+1})) = c_{i+1}(t_{i+1}) \),
3. \( t_{i+1} = \min\{t \in I_i, c_i(t) \in G_i\} \).

The initial condition for the hybrid execution is \( c_0(t_0) \).

Dynamical systems Let \( q \in \mathcal{Q} \) be the configuration space of a mechanical system.\(^1\) We will consider Lagrangians, \( L : T\mathcal{Q} \to \mathbb{R} \), describing mechanical or robotic systems, which are of the form

\[ L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} - V(q), \]

with \( M(q) \) the (positive definite) inertial matrix, \( \frac{1}{2} \dot{q}^T M(q) \dot{q} \) the kinetic energy and \( V(q) \) the potential energy. In this case, the Euler-Lagrange equations yield the (controlled) equations of motion for the system given in coordinates by:

\[ M(q) \ddot{q} + \dot{C}(q, \dot{q}) \dot{q} + N(q) = \mathcal{Y}, \]

where \( C(q, \dot{q}) \) is the vector of centripetal and Coriolis terms (cf. [20]), \( N(q) = \frac{\partial L}{\partial q^T} \) and \( \mathcal{Y} \) is the vector of torque inputs.

For an electrical system, the generalized coordinates are chosen as inductor currents: \( I_M^T = [i_1, i_2, \ldots, i_{n_M}] \) and capacitor voltages: \( v_E^T = [v_1, v_2, \ldots, v_{n_E}] \). Therefore, when an electrical system is included with a mechanical system (called the electromechanical system), we get the Extended Lagrangian, \( L_e : T\mathcal{Q}_e \to \mathbb{R} \), and is given by:

\[ L_e(q, \dot{q}, I_M, v_E) = \frac{1}{2} \dot{q}^T M(q) \dot{q} - V(q) + W_e(I_M, v_E, q), \]

where \( W_e(I_M, v_E, q) \) is the energy stored in the magnetic and electric fields of the system.

In this paper, we will consider a particular case of elecromechanical system which comprises of \( n_M \) permanent

\(^1\)For simplicity, in the models considered, we assume that the configuration space is identical to \( \mathbb{R}^n \).
magnet DC (PMDC) drives. It is apparent to choose this system since most of the electromechanical systems have motors as primary actuators. The experiment we conducted to validate Zeno behavior also has one (PMDC) motor as actuator. Hence \( W_c \) reduces to:

\[
W_c(t_m, \dot{q}) = \frac{1}{2} \dot{q}^T L_M t_M - K_w \cos(q) t_M .
\]  

(5)

Readers are encouraged to go through [28] for a detailed derivation of this realization. \( L_M \in \mathbb{R}^M \times \mathbb{R}^M \) is the inductance matrix \(^2\) and \( K_w \in \mathbb{R}^M \times \mathbb{R}^M \) is the diagonal matrix of motor constants of the motors. The resulting dynamics is given by:

\[
R_M t_M + L_M t_M + K_w \dot{q} = \dot{V}_M(q, \dot{q}),
\]  

(6)

where \( R_M \in \mathbb{R}^M \times \mathbb{R}^M \) is the resistance matrix, \( \dot{V}_M \in \mathbb{R}^M \), a function of position and velocity of the mechanical system, is the feedback control law input in the form of voltage. Also, the torque \( \Upsilon \), mentioned in 3, is the torque delivered by the motors and hence will be a function of current, \( t_M \):

\[
\Upsilon(t_M) = K_w \dot{t}_M ,
\]  

(7)

where \( K_w \in \mathbb{R}^M \times \mathbb{R}^M \) is the diagonal matrix of torque constants of the motors.

Defining the state of the system as \( x = (q, \dot{q}, t_M) \), the vector field, \( f_{L_e} \), associated with the extended Lagrangian \( L_e \) of the form 4, takes the following form:

\[
\dot{x} = f_{L_e}(x) = \begin{bmatrix}
\dot{q} \\
\dot{q} - M^{-1} (C(q, \dot{q}) \dot{q} + N(q) + K_w t_M)
\end{bmatrix} .
\]  

(8)

The readers should make note of the fact that the Lagrangian(not \( W_c(t_M, \dot{q}) \)) includes the mechanical dynamics of the rotors and gearboxes.

**Holonomic constraints:** We now define the holonomically constrained mechanical system with an Extended Lagrangian \( L_e \) and a holonomic constraint \( \eta : Q_e \rightarrow \mathbb{R} \). For such systems, the constrained equations of motion can be obtained from the equations of motion for the unconstrained system (3), and are given by (cf. [20])

\[
M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + N(q) = d \eta(q)^T \lambda + \Upsilon,
\]  

(9)

where \( \lambda \) is the Lagrange multiplier which represents the contact force and \( d \eta(q) = \left( \frac{\partial \eta}{\partial q}(q) \right)^T \). For the reduced electromechanical system that we are considering (of the form 5), \( \Upsilon \) reduces to the form 7.

The constraints that we consider in this paper are only mechanical constraints and not electrical constraints. From the constrained equations of motion (9), for \( x = (q, \dot{q}, t_M) \), we get the vector field (see [20]):

\[
\dot{x} = f_{L_e}^H(x) = f_{L_e}(x) + \begin{bmatrix}
0 \\
M(q)^{-1} d \eta(q)^T \lambda(q, \dot{q})
\end{bmatrix} .
\]  

(10)

Note that here \( f_{L_e}^H \) is a vector field on the manifold \( T_{\eta^{-1}(0)} \). For this dynamical system, solutions stay on the constraint surface \( \eta^{-1}(0) \) as long as the constraint force \( \lambda \) is positive. We call a solution \( \chi \) to \( f_{L_e}^H \) on the interval \( \bar{I} = [\bar{t}_0, \bar{t}_j] \) a constrained execution and denote it by \( \dot{\chi} = \bar{I}(\bar{t}, \bar{\epsilon}) \).

**Unilateral Constraints:** The domain, guard and reset map (or impact equations) will be obtained from unilateral constraint \( h : Q_e \rightarrow \mathbb{R} \) which gives the set of admissible configurations of the system; we assume that the zero level set \( h^{-1}(0) \) is a smooth manifold.

Define the domain and guard, respectively, as:

\[
D_h = \{(q, \dot{q}, t_M) \in T Q : h(q) \geq 0 \},
\]  

(11)

\[
G_h = \{(q, \dot{q}, t_M) \in T Q : h(q) = 0 \text{ and } dh(q) \dot{q} \leq 0 \}.
\]

The reset map associated to a unilateral constraint is obtained through impact equations of the form (see [6], [19]):

\[
R_h(q, \dot{q}, t_M) = \begin{bmatrix}
\dot{q} - (1 + \varepsilon) \frac{dh(q)}{dh(q)} M(q)^{-1} dh(q)^T \\
\bar{t}_M
\end{bmatrix} .
\]  

(12)

Here \( 0 < \varepsilon < 1 \) is the coefficient of restitution. This reset map corresponds to rigid-body collision under the assumption of frictionless impact. Examples of more complicated collision laws that account for friction can be found in [6] and [26].

**Definition 3:** A simple electromechanical hybrid Lagrangian (or hybrid extended Lagrangian) is defined to be a tuple \( L_e = (Q_e, L_e, h) \), where

- \( Q_e \) is the configuration space (assumed to be \( \mathbb{R}^{n_e + n_M} \)),
- \( L_e : T Q_e \rightarrow \mathbb{R} \) is an extended Lagrangian of the form (4),
- \( h : Q_e \rightarrow \mathbb{R} \) is a unilateral constraint.

Given a hybrid extended Lagrangian \( L_e = (Q_e, L_e, h) \), associated is the simple electromechanical hybrid system (SEHS):

\[ \mathcal{X}_{L_e} = (D_h, G_h, R_h, f_{L_e}). \]

If the electromechanical system were to be eliminated from the Hybrid system, then \( \mathcal{X}_{L_e} \) becomes a Lagrangian Hybrid system consisting of only the dynamics of Lagrangian systems.

**III. Zeno Behavior**

We now introduce Zeno behavior and the corresponding notion of Zeno equilibria, and we consider the stability of these equilibria. We will present a theorem that from the literature characterizing when there exists Zeno behavior (or stable Zeno equilibria) in hybrid mechanical systems. This theorem is very important in that if stable Zeno equilibria exist, the hybrid system must be completed.

**Definition 4:** An execution \( \chi^\infty \) is Zeno if \( \Lambda = \mathbb{N} \) and

\[
t_\infty := \lim_{k \rightarrow \infty} t_k - t_0 = \sum_{k=0}^{\infty} t_{k+1} - t_k < \infty .
\]

\(^3\)Again, for \( n_L \) DC motors alone, \( n_C = 0 \), implying \( Q_e \in \mathbb{R}^{n_L} \).
Here $t_∞$ is called the Zeno time.

**Zeno Equilibria** If $Ψ^{IH_{Le}}$ is a Zeno execution of a SEHS, $Ψ^{IH_{Le}}$, then its Zeno point is defined to be

$$x_∞ = (q_∞, ̇q_∞, t_{M_∞}) = \lim_{k→∞} (q_k(t), ̇q_k(t), t_{M_k}(t_k)).$$  \hspace{1cm} (13)

These limit points are intricately related to a type of equilibrium point that is unique to hybrid systems: Zeno equilibria.

**Definition 5:** A Zeno equilibrium point of a SHS $Ψ^{H}$ is a point $x^* ∈ C$ such that $R(x^*) = x^*$ and $f(x^*) ≠ 0$.

If $Ψ^{IH_{Le}}$ is a SEHS, then due to the special form of these systems we find that the point $(q^*, ̇q^*, t^*_{M})$ is a Zeno equilibrium point iff $̇q^* = R_h(q^*, ̇q^*, t^*_{M})$, with $R_h$ given in (12). In particular, the special form of $R_h$ implies that this holds iff $dh(q^*) ̇q = 0$. Therefore the set of all Zeno equilibria for a SEHS is:

$$Z = \{ (q, ̇q, t_{M}) ∈ D_h : h(q) = 0, \quad dh(q) ̇q = 0 \quad \text{and} \quad f_{Le}(q, ̇q, t_{M}) ≠ 0 \}. \hspace{1cm} (14)$$

Note that if $dim(Q_e) > 1$, the Zeno equilibria in Lagrangian hybrid systems are always non-isolated.

The following theorem, which extends the results of [13], [14], [15] to simple electromechanical hybrid systems, provides sufficient conditions for existence of Zeno executions in the vicinity of a Zeno equilibrium point. Space constraints prevent an inclusion of the proof, but it is rather straightforward extension of the results of [13], [15].

**Theorem 1:** Let $Ψ^{H_{Le}}$ be a simple electromechanical Lagrangian hybrid system and let $x^* = (q^*, ̇q^*, t^*_{M})$ be a Zeno equilibrium point of $Ψ^{H_{Le}}$. Then the double derivative of the constraint function will result in:

$$\ddot{h}(q^*, ̇q^*, t^*_{M}) = (̇q^*)^T H(h(q^*)) ̇q + h(q^*) M(q^*)^{-1} \left( -C(q^*, ̇q^*) ̇q - N(q^*) + K_p t_{M} \right). \hspace{1cm} (15)$$

where $H$ is the hessian of $h$ at $q$.

If $0 ≤ ε < 1$ and $\ddot{h}(q^*, ̇q^*, t^*_{M}) < 0$, there exists a neighborhood $W ⊂ D_{Le}$ of $(q^*, ̇q^*, t^*_{M})$ such that for every $(q_0, ̇q_0, t_{M_0}) ∈ W$, there is unique Zeno execution $χ$ of $Ψ^{H_{Le}}$ with $c_0(t_0) = (q_0, ̇q_0, t_{M_0})$.

This theorem is essential to this paper because, if a system is determined to be Zeno through these conditions, it is necessary to complete this system to allow solutions to be carried past Zeno points.

**IV. COMPLETED HYBRID SYSTEMS**

Using the notion and concepts used thus far, we now introduce the notion of a completed simple electromechanical hybrid Lagrangian system. Loosely speaking, a completed hybrid system consists of hybrid dynamics and constrained dynamics, with transitions between these two types of dynamics (see Fig. 1 for a graphical representation of a completed hybrid system). The idea is that, if the system has stable Zeno equilibria it evolves according to the hybrid dynamics until the Zeno point is reached, at which time a transition to the constrained dynamics is made.

Formally, completed hybrid systems have been defined in the following manner [4], [5], [18], [21], [22], [23] (and are often termed complementary Lagrangian hybrid systems); if $L_c$ is a simple hybrid extended Lagrangian and $Ψ^{IH_{Le}}$ the corresponding SEHS, the corresponding completed extended Lagrangian hybrid system$^4$ is:

$$\overline{Ψ}^{IH_{Le}} = \{ \begin{array}{ll}
\mathcal{D}_h & \text{if } h(q) = 0, \; dh(q) ̇q = 0, \\
\mathcal{D}_{Le} & \text{otherwise}
\end{array}$$

where $λ$ is the Lagrange multiplier obtained from $h$ viewed as a holonomic constraint. That is, beginning with a simple Lagrangian hybrid system $Ψ^{H_{Le}}$, as the execution converges toward the Zeno point, $h → 0$. This implies that after the Zeno point is reached, there should be a switch to a holonomically constrained dynamical system with holonomic constraint $η = h$. Letting $\mathcal{D}_h = (Z, f^L_h)$ be the dynamical system obtained from this unilateral constraint as in (9) with $Z$ the set in (14), the system will evolve according to this dynamical system until the constraining force is “released” which is detected through a change in sign of the Lagrange multiplier. At this point, the system switches back to the original hybrid system until another Zeno point is reached.

We can consider solutions to completed hybrid systems by concatenating solutions to its individual components. Intuitively, a solution to a completed hybrid system consists of unconstrained motion, followed by constrained motion (when the Zeno point is reached), followed again by unconstrained motion (when the Lagrange multiplier changes sign). This idea is made precise in the following definition (recall that both executions and constrained executions were introduced in Subsection II-A):

**Definition 6:** Given a completed system $\overline{Ψ}^{IH_{Le}}$, a completed execution is $\overline{χ}$ of $\overline{Ψ}^{IH_{Le}}$, a sequence of alternating hybrid and constrained executions of $\overline{χ} = \{\overline{χ}^{(1)}, \overline{χ}^{(2)}, \overline{χ}^{(3)}, \overline{χ}^{(4)}, \ldots\}$ that satisfies the following conditions:

(i) For each pair $\overline{χ}^{(i)}$ and $\overline{χ}^{(i+1)}$:

$$τ^{(i)}_∞ = τ^{(i+1)}_0 \quad \text{and} \quad c_∞^{(i)} = c^{(i+1)}_0 \left( z^{(i+1)}_0 \right)$$

(ii) For each pair $\overline{χ}^{(i)}$ and $\overline{χ}^{(i+1)}$:

$$\overline{χ}^{(i)}_j = \overline{χ}^{(i+1)}_j \quad \text{and} \quad c^{(i)}_j = c^{(i+1)}_0 \left( z^{(i+1)}_0 \right)$$

$^4$As was originally pointed out in [4] for completed extended Lagrangian hybrid system, this terminology (and notation) is borrowed from topology, where a metric space can be completed to ensure that “limits exist.”
where the superscript $(i)$ denotes the values corresponding to the $i^{th}$ execution $\mathcal{X}_i$ or $\mathcal{Z}_i$, with $t_{\infty}^{(i)}$, $\epsilon_{\infty}^{(i)}$ denoting the Zeno time and Zeno point in the case when the $i^{th}$ execution is a Zeno execution $\mathcal{X}_i$.

**Zeno periodic orbits** A Zeno periodic orbit is a completed execution $\mathcal{X}$ with initial condition $\epsilon^{(1)}(0) = x^*$ that satisfies $\epsilon_{\infty}^{(2)} = \epsilon_{\infty}^{(3)}(1) = x^*$. The period of $\mathcal{X}$ is $T = t_{\infty}^{(2)} = t_{\infty}^{(3)}$. In other words, this orbit consists of a constrained execution starting at $x^*$, followed by a Zeno execution with infinite number of non-plastic impacts, which converges in finite time back to $x^*$. In the special case of plastic impacts $\epsilon = 0$, a periodic orbit is called a simple periodic orbit.

**Simulating completed hybrid systems.** Due to the fact that completed hybrid systems have Zeno executions, and because it is not possible to compute the entirety of these executions, a procedure must be given to simulate completed hybrid systems. Such a procedure is developed formally in [22],[23]), but for the purposes of this paper, we only discuss the practical aspects of this approach. First, a hybrid execution is simulated, until it reaches an impact at some time $t_i$, with the state $(q(t_i), \dot{q}(t_i), t_M)$ satisfying $|h(q(t_i))| < \delta$ with $\delta > 0$ a sufficiently small simulation parameter. (This implies that the execution is “close” to the Zeno point which satisfies $h(q(t_i), \dot{q}(t_i), t_M(t_i)) = 0$.) When this condition is satisfied, the hybrid execution is truncated and the algorithm applies a reinitialization map $(q^*, \dot{q}^*, t_M^*) = R^t(q(t_i), \dot{q}(t_i), t_M(t_i))$, with $R^t$ the reset map given in (12) with $\epsilon = 0$, i.e., it applies a perfectly plastic impact. This guarantees that $(q^*, \dot{q}^*, t_M^*)$ is a Zeno equilibria or $(q^*, \dot{q}^*, t_M^*) \in Z$. At this point, the constrained dynamics (10) are simulated with $(q^*, \dot{q}^*, t_M^*)$ as an initial condition. If it is detected that $\lambda = 0$ the simulation switches back to the hybrid system and the process is repeated.

**V. Modeling the Double Pendulum with a Mechanical Stop**

We now consider the hybrid system model of the physical “Zeno system” that will be used. In particular, a double pendulum with a mechanical stop and with the top link being controlled by a PMDC motor (see Fig 2a). The goal of this section is to discuss how the system is modeled as a hybrid system, show formally that the system has Zeno behavior, use this knowledge to complete the hybrid system model and finally simulate the system. In the end we find that the simulated system has a Zeno periodic orbit. It is important to note that the analysis done in this section is much like what any researcher would do studying hybrid systems with Zeno behavior. The next section will show that, in fact, all of the theory utilized presents an accurate model for the system in reality.

Consider a double pendulum with a mechanical stop (Fig. 2a). This system has rigid links link1 and link2 of lengths $L_1, L_2$ and masses $m_1, m_2$, respectively, which are attached by a passive joint. The links are assumed to have uniform mass distribution. Link1 is actuated by a permanent magnet DC motor for controlling the trajectories (see fig. 5b). In this model the masses of the first link ($m_{1L}$) and the rotating parts (armature and gear box) of the motor ($m_{m}$) are included together and denoted as $m_1 = m_{1L} + m_m$, while the mass of the second link is denoted as $m_2 = m_{2L}$. The resulting shift in the center of mass is also included while computing the moments of inertia. This ensures that all the attributes of the physical model are also included in the mathematical model.

To construct the hybrid system model for the double pendulum, we begin by considering the hybrid extended Lagrangian: $$L_{P_e} = (Q_{P_e}, L_{P_e}, h_{P_e}),$$ where $Q_{P_e}$ is the configuration space spanned by $q = (\theta_1, \theta_2, t_m)$, where $\theta_1$ is the angle between link1 and vertical line from top end of link1 to ground (see fig. 2a), $\theta_2$ is relative angle between link1 and link2 (constrained to be positive), and $t_m$ is the motor current. $L_{P_e}$ is the extended Lagrangian for the electromechanical system (given in Fig. 2a), which thus has the standard form given in (4). The unilateral constraint $h_{P_e}$ describes the constraint on link2, i.e., it is not allowed to pass through the mechanical stop, and is thus given by: $$h_{P_e}(q) = \theta_2 \cdot b$$

Now we obtain a hybrid model from this hybrid extended Lagrangian in almost the exact way as described in Subsection II-A. In particular, the state-space of the electromechanical system is given by $\mathbb{R}^5$ and spanned by $(q, \dot{q}, t_m) = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, t_m)$. From the hybrid extended Lagrangian $L_{P_e}$ we obtain a simple hybrid system given by: $$\mathcal{FE}_{P_e} = (D_{P_e}, G_{P_e}, R_{P_e}, f_{P_e}).$$

The domain and guard are given as in (11) since the unilateral constraint is not a function of the current. In particular, the guard $G_{P_e}$ is the subset of domain $D_{P_e}$ where link2 is “locked” to mechanical stop. The reset map, $R_{P_e}(q, \dot{q}, t_m)$ is given as in (12). Finally, the vector field $f_{P_e}$ is an extended Lagrangian vector field of the form (8) with the vector $t_M$ having only one motor, $t_m$.

Torque is controlled indirectly by varying the voltage input to motor. A simple P-D control law is adopted with $\theta_1, \dot{\theta}_1$ being the inputs: $$V_m(q, \dot{q}) = -K_p \theta_1 - K_d \dot{\theta}_1,$$ (17) $K_p$ and $K_d$ are proportional and derivative constants.
Formally verifying Zeno behavior. We now verify that the hybrid system model for the double pendulum with a mechanical stop displays Zeno behavior by checking the conditions of Theorem 1. This is an important step in the simulation process, because if the model has stable Zeno equilibria it implies that it will display Zeno behavior for a non-trivial set of initial conditions. Therefore, the model must be completed to allow solutions to be taken past the Zeno points.

For the double pendulum, the set of Zeno equilibria is:

$$ Z_P = \{ (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, i_m) \in D_P : \theta_1 = 0, \dot{\theta}_2 = 0, f_P \neq 0 \}, $$

that is, the set of Zeno equilibria are the set of points where the lower link is “locked”. Taking the second derivative of the unilateral constraint $h_P(q, \dot{q}, i_m)$ leads to $\dot{h}_P(q, \dot{q}, i_m) = \dot{\theta}_2$. Therefore as long as $\dot{\theta}_2 < 0$ immediately after every impact the system is Zeno stable (per Theorem 1). We thus need to find the conditions on the configuration of the system where this inequality holds. In particular $\dot{\theta}_2(t)$ can be obtained from the the vector field (8): $\dot{\theta}_2(t) = (f_P(x))_{\dot{x}}$. Due to the complexity of the model being considered, it is not possible to simply state this expression in symbolic form. But, for the double pendulum considered for the experiment, with all the physical parameters substituted, $\dot{h}$ is found to be:

$$ \dot{h}(q, \dot{q}, i_m) = -(2.92437)i_m - (27.5697)\sin(\theta_1). $$

The blue region in the figure on the right indicates where $\dot{h} < 0$. It can be inferred from this figure that the stable Zeno equilibria are essentially the set of Zeno equilibria where $\dot{\theta}_1$ is positive, i.e., where the pendulum is swinging “to the right.” This is a large set of configurations, so the double pendulum with a mechanical stop is Zeno and it is necessary to complete this hybrid system.

The analysis of proving the existence of Zeno behavior now motivates the introduction of completed hybrid systems since we will have stable Zeno equilibria at a large collection of points. Now the system can be taken past the Zeno point, which basically means that the double pendulum will “lock” after Zeno execution. That is, we obtain a vector field, $f_P^\eta$, for the constrained system (in better terms “locked system”), which is given as in (10) with $\eta = \lambda$, the Lagrange multiplier obtained from the unilateral constraint $h_P$. Thus, as outlined in Section III, the completed double pendulum system is given as in (16) by $\mathcal{H}_{L_1}$, where the $\mathcal{H}_{L_1}$ is the “constrained” system with dynamics given by $f_P^\eta$ corresponding to the pendulum being “locked” and $\mathcal{H}_{L_2}$ is the “unconstrained system” corresponding to the pendulum in “unlocked” position. We can thus simulate this resulting complete hybrid system through the methods discussed in Section III.

In this case, we enforce the holonomic constraint lambda for completion. So $\eta = \lambda$.

Simulating the Double Pendulum Model. Fig. 3 shows a CAD model of the double pendulum considered for the experiment. We began, with the list of parts and data sheets for these components, by estimating the parameters of the physical system. Yet, even with the data sheets, it was not possible to estimate all of these values accurately. For example, the resistance of the circuit, which have MOSFETs (Metal Oxide Semi-conductor Field Effect Transistor) for switching the H-bridge, must be taken into account when determining the parameters of the system. As a result, and coupled with a detailed Solidworks model (Fig. 3), we are able to determine the physical parameters of the system used in simulation, which represents our attempt to accurately represent the physical parameters of the system shown in Fig. 5.

From the estimated physical parameters for the system, we are able to simulate the double pendulum. Since the goal is to validate Zeno behavior as a modeling paradigm, we looked for control gains that resulted in a Zeno periodic orbit in the completed hybrid system. In particular, we found that for $K_p = 2.5$ and $K_d = -1$ the end result is a Zeno periodic orbit, which can be seen in Fig. 4 which shows the phase portraits for this orbit. The top of the $(\theta_1, \dot{\theta}_1)$ phase portrait shows jumps due to the presence of impacts of link2 with link1. Same is true with the second phase portrait, $(\theta_2, \dot{\theta}_2)$, with jumps being seen on the left side. The impacts die down once the solution reaches the Zeno point and then the pendulum resumes normal constrained motion. This cycle repeats with alternating phases of constrained and unconstrained motions, indicating that it is a Zeno periodic orbit. The goal is to show that this simulated behavior correctly predicts the behavior of the physical system.

VI. EXPERIMENTAL RESULTS

This section discusses an experiment conducted on a physical double pendulum with a mechanical stop as shown
in Fig. 5. The goal is to run this physical system with the same controllers as those that were established in the previous section to show that, in fact, the simulation of this system captures its physical behavior (especially with respect to Zeno behavior, completion, and the existence of a Zeno periodic orbit).

To run the experiment, a ball with a given coefficient of restitution is placed at the mechanical stop; in this case a coefficient of restitution of $\varepsilon = 0.2$, which includes both the energy lost in the ball and the gear train impacts. The system was then run with the same P-D control law as the simulated system. Note that the mass of the bottom link, link2, had to be increased from 1.61 kg to 2.92 kg in order to see pronounced impacts due to the low coefficient of restitution (as illustrated in Fig. 5b). The end result is a very close agreement with the simulated behavior of the system as can be seen in Fig. 6, indicating that Zeno behavior provides a valid approximation of physical phenomena. Link to the video comparing real and simulated behavior is given in [1]. Of special interest is the fact that simulation predicted the existence of a Zeno periodic orbit, and we find that the physical system in fact displays a Zeno periodic orbit (or a physical approximation thereof). To better understand this comparison between real and simulated behavior, we discuss the plots in Fig. 6.

Figure 6a shows a comparison of simulated and physical behaviors over time with the periods of constrained and unconstrained motions indicated. In the lower waveform, when $\theta_2 > 0$ the system evolves according to the hybrid system $\mathcal{H}_P$, until the Zeno point is reached, i.e., $\theta_2 = 0$, or link1 is “locked” to link2. At this point, the system evolves under the constrained dynamics, until the Lagrange multiplier changes sign and the bottom link is released. Figure 6b zooms into one period of the Zeno periodic orbit consisting of a Zeno solution, followed by a constrained phase, followed by release; the simulated and physical behavior are compared in this figure. One can see that there is very good agreement between the predicted and actual behavior. In particular, the simulation accurately models the first large impact in the system, and the constrained period in simulation approximates small oscillations in the physical system as a result of vibrations in the ball when link2 is in contact with link1.

The phase portraits of the simulated and physical system are compared in Fig. 6c; again, the simulated system has a Zeno periodic orbit and we find that the physical system also displays a “Zeno periodic orbit” in the sense that the phase portrait is periodic with phases of constrained and unconstrained motion, with transitions to the constrained phase occurring at the Zeno point and transitions to the unconstrained phases occurring when the link2 is released. Note that the largest deviations for the physical and simulated system don’t occur near the impacts and Zeno points, but are rather due to time delays in the change of motor direction at the apex of the pendulum motion; a delay that the simulated system was not able to completely capture. The behavior of the simulated vs. the physical system near the Zeno point can be seen in Fig. 6d. Here one can see very good agreement between the the predicted and actual behavior. The physical system clearly has an accumulation point at the Zeno equilibria just as the theory predicted.

VII. Conclusion

This paper showed that Zeno behavior, while it may not “exist” in reality, provides an accurate model of real physical phenomena. Moreover, all of the theory that has been proven of the years with respect to Zeno behavior is practically useful in predicting the behavior of physical systems. In particular, we utilized the notions of extended Lagrangians, Zeno equilibria, hybrid system completion, and Zeno periodic orbits. The existence of these theoretical constructs were used to properly simulate the Zeno system modeling a double pendulum with a mechanical stop. A physical version of this system was built, and the same controller applied to the simulated system was applied to this physical system. The end result was very good agreement between the simulated and physical behavior. This provides evidence for the claim that Zeno behavior provides a good approximation to phenomena that can occur in physical systems. As such, studying this behavior is an important research direction in hybrid systems.
Fig. 6: The simulated vs. physical behavior of a ball with a coefficient of restitution of 0.2.