Sufficient Conditions for the Lipschitz Continuity of QP-based Multi-Objective Control of Humanoid Robots

Benjamin Morris, Matthew J. Powell, Aaron D. Ames

Abstract— **In this paper we analyze the continuity properties of feedback controllers that are formulated as state-dependent quadratic programs (QP), with specific application to motion control for humanoid robots. With a desire to achieve multiple simultaneous goals in locomotion and manipulation, we develop a generalized QP-based control law through the use of multiple control Lyapunov functions (CLFs). Motivated by simulation studies showing cases where QP-based control loses Lipschitz continuity, the main result of this paper is a set of sufficient conditions under which such continuity properties will hold. This result provides conditions under which any number of tasks encoded as CLFs can be simultaneously exponentially stabilized. Finally, these results are demonstrated in a simulation of a simple humanoid robot climbing a vertical ladder.**

I. Introduction

The control of robotic systems often involves the need to operate at or near a robot's limits of performance. Most often this means executing a single task in some limiting case, such as walking or running as fast as possible, achieving the largest stability margin for a given gait, operating at maximum efficiency, etc. Alternatively, limits of performance might also be reached by executing many relatively simple tasks simultaneously, such as walking while simultaneously reaching with both hands under constraints on torque saturation, energy consumption, and center-of-mass accelerations. When objectives like this arise Model Predictive Control (MPC) or online Quadratic Programs (QP) can be used to explicitly incorporate a wide variety of objectives and constraints including saturation, quantization, power limits, etc., enabling feedback controllers to operate very near the limits of performance. This is one reason feedback controllers based on online optimization are gaining popularity in the robotics community [1], [22], [21]. Controllers based on control Lyapunov functions (CLFs), enforced via online QPs, are used for locomotion in [3], [4], [11] and the simultaneous control of locomotion and manipulation in [5].

In the process of further generalizing the class of QP-based controllers we have observed a number of cases where the resulting feedback controller exhibited rapid chatter (suggesting that the feedback was non-Lipschitz) even in apparently simple cases such as standing. Motivated by these case studies, this paper provides a set of sufficient conditions under which QP-based control for humanoid robots will be Lipschitz continuous. In deriving these conditions we find that the relaxation factors used in [5] are not necessary to achieve locally exponentially stabilizing controllers.

In developing sufficient conditions for the Lipschitz continuity of QP-based control we make use of a number of foundational results in the literature. Robinson's conditions of constraint regularity [16], [17] and the earlier conditions of Mangasarian and Fromovitz [13] predict when a QP will remain solvable under small perturbations in its data. Davidson's subsequent analysis of Lipschitz continuity of the extreme points of the feasible set [8] and reformulation of the MF regularity conditions provide a numerically convenient method of determining Lipschitz continuity of the feasible set. A thorough summary of results in the continuity and Lipschitz continuity of QPs and LPs is reviewed in [7] and the references therein. Additional studies involving continuity properties of linear and quadratic programming are available in [2], [15]. In general, conditions of continuity for constrained QP-based control will differ from those of infinite- or finite- horizon constrained optimal controllers. Conditions of Lipschitz continuity for more traditional optimal control problems are reviewed in [10].

The remainder of the paper is organized as follows: In Section II a feedback law is formulated as a state-dependent QP. In Section III two main results are presented. The first is a theorem stating sufficient conditions under which the minimizer of a state-dependent QP is both unique and Lipschitz continuous with respect to the state. A metric is provided to measure how 'close' the QP is to losing these properties. The second main result is a corollary giving conditions under which simultaneous tasks (encoded as equilibrium points or periodic orbits) can be simultaneously stabilized. A set of simulation studies is presented in Section IV, with conclusions drawn in Section V.

II. Motivation–CLF Based QPs

Given the goal of simultaneously achieving a set of tasks, each represented as zeroing a vector of output functions, this section develops a control Lyapunov function (CLF) based Quadratic Program (QP). If the resulting QP has a unique, Lipschitz continuous solution, then when implemented as a feedback control law [3], [4], [11], the QP will simultaneously stabilize the set of tasks.

A. System Model

Let $Q \subset \mathbb{R}^n$ be the configuration space of a robot consisting of generalized coordinates $q \in Q$. The equations

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of motion for a robot are given in the general form by the Euler-Lagrange formula:

$$D(q)\ddot{q} + CG(q,\dot{q}) = B\tau \tag{1}$$

where D is the inertia matrix, CG is a vector containing the Coriolis and gravity terms, and B is the actuation matrix which determines the way in which the torque inputs, $\tau \in \mathbb{R}^m$, actuate the system. We will assume that the equations of motion correspond to the "unpinned" model of the robot, so that we can explicitly use holonomic constraints to describe the interaction of the robot with the environment.

Consider a vector of holonomic constraints: $h(q) = 0$, with $h(q) \in \mathbb{R}^p$. Defining the Jacobian $J_h(q) = \frac{\partial h(q)}{\partial q}$, the holonomic constraints are enforced through constraint (or contact) forces $F \in \mathbb{R}^p$ which appear in the constrained dynamics:

$$D(q)\ddot{q} + CG(q,\dot{q}) = B\tau + J_h(q)^T F \tag{2}$$

For the above dynamics to be valid, the torques τ, constraint forces F, and resulting accelerations $\ddot{q}$ must satisfy the following:

$$\dot{J}_h(q,\dot{q})\dot{q} + J_h(q)\ddot{q} = \mathbf{0}_{p\times 1} \tag{3}$$

$$RF > \mathbf{0}_{p\times 1}, \tag{4}$$

where R is a constant $(p \times p)$ matrix capturing friction and directional constraints on the generalized reaction forces.

With a view toward formulating the above constraints in a QP framework, the equations of motion (2) can be written:

$$D(q)\ddot{q} + CG(q,\dot{q}) = \underbrace{\begin{bmatrix} B & J_h^T(q) \end{bmatrix}}_{B_h(q)} \underbrace{\begin{bmatrix} \tau \\ F \end{bmatrix}}_{u}, \tag{5}$$

with $u \in \mathbb{R}^{m+p}$. This allows for the formulation of an affine control system of the form:

$$\begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = f(q,\dot{q}) + g(q)u, \tag{6}$$

where

$$f(q,\dot{q}) = \begin{bmatrix} \dot{q} \\ -D^{-1}(q)CG(q,\dot{q}) \end{bmatrix}, \quad g(q) = \begin{bmatrix} \mathbf{0}_{n\times(m+p)} \\ D^{-1}(q)B_h(q) \end{bmatrix}. \tag{7}$$

The reaction force constraints (3) and (4) will be combined with additional equalities and inequalities in Section II-C.

B. Tasks and Control Lyapunov functions

Suppose that we wish to accomplish K tasks, with each task can be expressed as $y_k(q) \in \mathbb{R}^{n_k}$, $n_k > 0$, $k \in \{1,\ldots,K\}$. For example, if the goal is to move a robot's hand to a desired location in Cartesian space, $(p_x^*, p_y^*)^T$, the corresponding output would be $y_k(q) = (p_x(q), p_y(q))^T - (p_x^*, p_y^*)^T$ where $(p_x(q), p_y(q))^T$ is the Cartesian position of the hand as calculated from the generalized coordinates. Output representations of tasks in the context of locomotion are prevalent and detailed constructions can be found in [23].

Remark 1: Note that tasks are often specified in terms of Jacobians [12], [19]. In the notation of this paper, the Jacobian for an individual task is $J_k = \frac{\partial y_k(q)}{\partial q}$. In the context of controlling multiple tasks, the typical approach is to prioritize the tasks, wherein the priority task is guaranteed to converge. Subsequent lower priority tasks are projected to the null-space of the higher priority tasks and, as a result, are not guaranteed to converge. It will be shown later that the formulation of tasks as outputs, coupled with the CLF based QP, allows for the simultaneous convergence of multiple tasks even in the case when they compete. ⌟

For any given task $y_k, k \in \{1,\ldots,K\}$ we will construct a corresponding control Lyapunov function that (under conditions to be presented) will guarantee exponential convergence of the task: $y_k \to 0$. This is accomplished by first considering the output dynamics. Since the outputs being considered are only functions of the configuration of the robot, differentiating twice yields:

$$\ddot{y}_k = \underbrace{\mathcal{L}_f^2 y_k(q,\dot{q})}_{(\mathcal{L}_f^2)_k} + \underbrace{\mathcal{L}_g \mathcal{L}_f y_k(q,\dot{q})}_{\mathcal{A}_k} u \tag{8}$$

with $\mathcal{L}$ representing the Lie derivative. Within the framework of input/output (IO) feedback linearization, one would check that the system has the same number of inputs as outputs and verify that each y_k satisfies a vector relative degree condition (typically vector relative degree 2) implying that the decoupling matrix $\mathcal{A}_k$ is well-defined and nonsingular. In the case of vector relative degree 2 this allows for IO feedback linearization through the choice of control law:

$$u = \mathcal{A}_k^{-1}(-(\mathcal{L}_f^2)_k + \mu_k) \tag{9}$$

resulting in $\ddot{y}_k = \mu_k$. Typically, one would then choose μ_k so that the resulting output dynamics are stable. One such feedback is the min-norm control law, constructed as follows: Let $\eta_k = (y_k, \dot{y}_k)$, so that our system can be equivalently expressed as the linear control system:

$$\dot{\eta}_k = \underbrace{\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}}_{F_k} \eta_k + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{G_k} \mu_k. \tag{10}$$

In the context of this control system, we can consider the continuous time algebraic Riccati equations (CARE):

$$F_k^T P_k + P_k F_k - P_k G_k G_k^T P_k + Q_k = 0 \tag{11}$$

with solution $P_k = P_k^T > 0$. The motivation for considering the CARE is that the control law $\mu_k = -G_k^T P_k \eta_k$ drives $y_k \to 0$ while minimizing the cost function:

$$\mathrm{Cost}_{\mathrm{LQR}} = \int_0^\infty \left(\eta_k^T Q_k \eta_k + \mu_k^T \mu_k\right) dt.$$

One can use P_k to construct a rapidly exponentially stabilizing control Lyapunov function (RES-CLF) (as in [3], [4]) that can be used to exponentially stabilize the output dynamics at a user defined rate of ε. In particular, define

$$V_k^\varepsilon(\eta_k) := \eta_k^T \underbrace{M_k^\varepsilon P_k M_k^\varepsilon}_{P_k^\varepsilon} \eta_k, M_k^\varepsilon = \begin{bmatrix} \varepsilon I_{n_k \times n_k} & 0 \\ 0 & I_{n_k \times n_k} \end{bmatrix}$$

wherein it follows that:

$$\dot{V}_k^\varepsilon(\eta_k) = \mathcal{L}_{F_k} V_k^\varepsilon(\eta_k) + \mathcal{L}_{G_k} V_k^\varepsilon(\eta_k)\mu_k$$

with

$$\begin{aligned} \mathcal{L}_{F_k} V_k^\varepsilon(\eta_k) &= \eta_k^T (F_k^T P_k^\varepsilon + P_k^\varepsilon F_k)\eta_k \\ \mathcal{L}_{G_k} V_k^\varepsilon(\eta_k) &= 2\eta_k^T P_k^\varepsilon G_k. \end{aligned}$$

The RES-CLF formulation of a specific task, y_k, allows for the construction of a controller—termed the *min-norm* controller—expressed in terms of a quadratic program through the objective of minimizing μ_k while satisfying a constraint on the rate of convergence:

$$\mu_k^*(q,\dot{q}) = \underset{\mu_k \in \mathbb{R}^{n_k}}{\operatorname{argmin}} \quad \mu_k^T \mu_k \qquad \text{(Min-Norm)}$$
$$\text{s.t.} \quad \hat{A}_k^{CLF}(q,\dot{q})\mu_k \leq \hat{b}_k^{CLF}(q,\dot{q}) \quad (\text{CLF}_k)$$

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where

$$\begin{aligned} \hat{A}_k^{CLF}(q,\dot{q}) &= \mathcal{L}_{G_k} V_k^\varepsilon(\eta_k), \\ \hat{b}_k^{CLF}(q,\dot{q}) &= -\varepsilon\gamma V_k^\varepsilon(\eta_k) - \mathcal{L}_{F_k} V_k^\varepsilon(\eta_k) \end{aligned} \qquad (12)$$

and we used the fact that $\eta_k = (y_k(q), \dot{y}_k(q,\dot{q}))$. In the absence of additional constraints the control law μ_k^* is Lipschitz continuous [9], [20] and applying this control law via (9) to (6) guarantees exponential convergence of the output y_k.

Although the above constructions stabilize the selected vector of outputs y_k, this control law naturally makes no guarantees that *all* of the tasks, $y = \{y_k\}_{k \in \{1,\dots,K\}}$ can be simultaneously satisfied, since there will potentially be many more entries of the total outputs $y = \{y_k\}_{k \in \{1,\dots,K\}}$ than there are actuators. The above constructions must also be modified if we have more actuators than outputs. Ideally, we would like an analytical approach that can be used to draw conclusions about the stabilizability of tasks, with no restrictions on the number of tasks that can be considered. In the case of multiple tasks or when constraints such as saturation or ground reaction forces are included a more sophisticated QP is needed to ensure convergence and Lipschitz continuity. With this in mind and with a view toward control Lyapunov functions, we will take an alternate route.

C. CLF based QPs

Consider again the goal of simultaneously achieving K tasks encoded as a set of output vectors: $\{y_k\}_{k \in \{1,\dots,K\}}$. The control objective of simultaneously satisfying all of these tasks can naturally be incorporated into the constraints of a QP. In particular:

Cost: Motivated by the cost for (Min-Norm), $\mu_k^T \mu_k$, and noting that in the case of a set of outputs (9) can be written:

$$\underbrace{\begin{bmatrix} (\mathcal{A})_1 \\ \vdots \\ (\mathcal{A})_K \end{bmatrix}}_{\mathcal{A}} u = -\underbrace{\begin{bmatrix} (\mathcal{L}_f^2)_1 \\ \vdots \\ (\mathcal{L}_f^2)_K \end{bmatrix}}_{\mathcal{L}_f^2} + \underbrace{\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_K \end{bmatrix}}_{\mu} \qquad (13)$$

we have that: $\mu^T \mu = u^T \mathcal{A}^T \mathcal{A} u + 2(\mathcal{L}_f^2)^T \mathcal{A} u + (\mathcal{L}_f^2)^T \mathcal{L}_f^2$ Since the value of $\mu^T \mu$ is minimized when the control objective of driving all of the outputs y_k to zero is satisfied, it may be desirable to balance this objective with torque minimization: $u^T u$. This motivates the final form of the cost function:

$$\begin{aligned} C(q,\dot{q},u) &= u^T H(q,\dot{q}) u + 2c^T(q,\dot{q}) u && (14) \\ H(q,\dot{q}) &= (1-\alpha)\mathcal{A}^T \mathcal{A} + \alpha I && (15) \\ c^T(q,\dot{q}) &= (1-\alpha)(\mathcal{L}_f^2)^T \mathcal{A} && (16) \end{aligned}$$

where $0 \leq \alpha \leq 1$ is a real-valued parameter that allows for the weighting between the two cost objectives; $\alpha = 0$ only priorities the control objective, while $\alpha = 1$ only prioritizes the minimization of torque (and reaction forces).

Inequality Constraints: For each of the tasks we can obtain a RES-CLF V_k^ε, which in turn yields inequality constraints of the form (12). These inequalities were stated in terms of μ_k, yet noting (9) they can be converted to a form dependent on u and combined:

$$\begin{aligned} A^{\text{CLF}}(q,\dot{q}) &= \begin{bmatrix} A_1^{\text{CLF}}(q,\dot{q})\mathcal{A}_1 \\ \vdots \\ A_K^{\text{CLF}}(q,\dot{q})\mathcal{A}_K \end{bmatrix}, && (17) \\ b^{\text{CLF}}(q,\dot{q}) &= \begin{bmatrix} b_1^{\text{CLF}}(q,\dot{q}) - A_1^{\text{CLF}}(q,\dot{q})(\mathcal{L}_f^2)_1 \\ \vdots \\ b_K^{\text{CLF}}(q,\dot{q}) - A_K^{\text{CLF}}(q,\dot{q})(\mathcal{L}_f^2)_K \end{bmatrix}. \end{aligned}$$

In addition, the ground reaction force constraints on the system (4) yield:

$$A^F(q,\dot{q}) = \begin{bmatrix} \mathbf{0}_{p \times m} & -R \end{bmatrix}, \quad b^F(q,\dot{q}) = \mathbf{0}_{p \times 1}. \qquad (18)$$

Finally, we may wish to constrain the actuator torques, τ, not to exceed prespecified limits. This yields the torque saturation inequality constraints:

$$A^\tau(q,\dot{q}) = \begin{bmatrix} I_{m \times m} & \mathbf{0}_{m \times p} \\ -I_{m \times m} & \mathbf{0}_{m \times p} \end{bmatrix}, b^\tau(q,\dot{q}) = \begin{bmatrix} \tau_{\max} \mathbf{1}_{m \times 1} \\ -\tau_{\min} \mathbf{1}_{m \times 1} \end{bmatrix}$$

with $\tau_{\max}$ and $\tau_{\min}$ the maximum and minimum values of the motor torques.

Equality Constraints: With the goal of constraining the dynamics according to (3) we obtain the equality constraints:

$$\begin{aligned} A_{eq}^F(q,\dot{q}) &= J_h(q) D(q)^{-1} B_h(q) && (19) \\ b_{eq}^F(q,\dot{q}) &= J_h(q) D(q)^{-1} CG(q,\dot{q}) - \dot{J}_h(q,\dot{q})\dot{q}. \end{aligned}$$

Combining the constructed cost and constraints yields the CLF based QP:

$$u^* = \underset{u\in\mathbb{R}^{m+p}}{\operatorname{argmin}} \quad (1-\alpha)\left(u^T\mathcal{A}^T\mathcal{A}u + 2(\mathcal{L}_f^2)^T\mathcal{A}u\right) + \alpha u^T u \qquad \text{(CLF-QP)}$$

$$\begin{aligned} \text{s.t.} \quad & A^{\mathrm{CLF}}(q,\dot q)u \le b^{\mathrm{CLF}}(q,\dot q) && \text{(CLF)} \\ & A^F(q,\dot q)u \le b^F(q,\dot q) && \text{(Contact Forces)} \\ & A^\tau(q,\dot q)u \le b^\tau(q,\dot q) && \text{(Torque)} \\ & A^F_{eq}(q,\dot q)u = b^F_{eq}(q,\dot q) && \text{(Dynamics)} \end{aligned}$$

The formulation of the CLF based QP allows for an understanding of the importance of the continuity of QP based feedback control laws. In particular, if this QP yields a continuous solution it implies that all of the tasks $\{y_k\}_{k\in\{1,\ldots,K\}}$ can be simultaneously stabilized (as will be demonstrated in Corollary 1). Yet there are no guarantees thus far that the above QP will have a solution, or if it does that the solution will be Lipschitz continuous in the state. There are numerous practical situations in which continuity fails (as will be demonstrated in Sect. IV), thus motivating the main results of this paper.

III. Sufficient Conditions for Lipschitz Continuity of QP-based Feedback

In general, when using online QPs for control, we would like the resulting feedback to be unique (as opposed to set-valued) and Lipschitz continuous (avoiding chatter and preserving existence and uniqueness of solutions). In this section we will develop a set of sufficient conditions under which a unique Lipschitz continuous solution to (CLF-QP) must exist.

Let $\mathcal{X}$ be a state manifold on which the state-dependent QP is defined. In the case of the dynamic model of (1), the state manifold is $\mathcal{X} = \mathcal{Q}\times\mathbb{R}^n$. Define a multivalued mapping $u^* : \mathcal{X} \rightrightarrows \mathbb{R}^{m+p}$ as the state-dependent set of minimizers for the QP as shown below, where H, c, A, A_{eq}, b, and b_{eq} are continuous functions of the state x:

$$\begin{aligned} u^*(x) = \underset{u\in\mathbb{R}^{m+p}}{\operatorname{argmin}} \quad & u^T H(x) u + 2c(x)^T u \\ \text{s.t.} \quad & A(x)u \le b(x) \\ & A_{eq}(x)u = b_{eq}(x). \end{aligned} \tag{20}$$

Based on the Mangasarian Fromovitz regularity conditions [13], we define the width of a feasible set as the unique solution to the following Linear Program (LP):

$$\begin{aligned} \omega(x) = & \max_{(u,w)\in\mathbb{R}^{m+p+1}} w \\ \text{s.t.} & \begin{bmatrix} A(x) & \mathbf{1}_{n_b\times 1} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} \le b(x) \\ & \begin{bmatrix} A_{eq}(x) & \mathbf{0}_{n_{beq}\times 1} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = b_{eq}(x), \end{aligned} \tag{21}$$

where A, b, A_{eq}, and b_{eq} are derived from (CLF-QP), n_b is the number of inequality constraints and n_{beq} is the number of equality constraints. Note that for any value of the state x, the above LP will always have a unique, feasible solution. We can now state the main theorem:

Theorem 1: [Sufficient conditions for Lipschitz continuity of QP control] *Consider the QP* (20)*, and suppose that the following conditions hold at at a point $x_0 \in \mathcal{X}$:*

1) $\omega(x_0) > 0$
2) $A_{eq}(x_0)$ *has full row rank*
3) $A(x)$*,* $A_{eq}(x)$*,* $b(x)$*, and* $b_{eq}(x)$ *are Lipschitz continuous at* x_0
4) $H(x_0) = H^T(x_0) > 0$
5) $H(x)$ *and* $c(x)$ *are Lipschitz continuous at* x_0

Then, the feedback $u^*(x)$ *defined in* (20) *is unique and Lipschitz continuous w.r.t. the state at* x_0.

The main result above establishes conditions for the simultaneous exponential stabilization of multiple tasks. To apply the theorem, consider again the goal introduced in Section II of stabilizing multiple tasks: $\{y_k\}_{k\in\{1,\ldots,K\}}$. For each of these tasks, there exists a coordinate transformation [18] that allows the dynamics (6) to be written in the form:

$$\begin{aligned} \dot\eta_k &= f_k(\eta_k, z_k) + g_k(\eta_k, z_k)u \\ \dot z_k &= q_k(\eta_k, z_k), \end{aligned} \tag{22}$$

where $\eta_k = (y_k, \dot y_k)$ are the output coordinates and z_k are normal coordinates to the output coordinates. In addition, we assume that $f_k(0, z_k) = 0$, i.e., we assume there exists a well-defined zero dynamics. In this case the zero dynamics surface $\mathcal{Z}_k$ defined by $\eta_k = 0$ is invariant and has dynamics $\dot z_k = q_k(0, z_k)$.

Let $\mathcal{O}_k$ be either an equilibrium point or periodic orbit of the zero dynamics $\dot z_k = q_k(0, z_k)$. In addition, assume that $\mathcal{O}_k \subset \mathcal{Z}_k$ is locally exponentially stable. Then the following result states that if the conditions in Theorem 1 hold, it is possible to simultaneously stabilize all of the sets $\mathcal{O}_k$ in the full-order dynamics.

Corollary 1: Assume the conditions of Theorem 1 hold for the QP (CLF-QP) *for all points in a neighborhood of $\mathcal{O}_k$ for all $k \in \{1,\ldots,K\}$. Then for the control law $u^*(x)$ obtained by solving* (CLF-QP)*, the sets $\mathcal{O}_k$ are locally exponentially stable in the full order dynamics* (6) *for all $k \in \{1,\ldots,K\}$.*

Remark 2: Following from the discussion in Remark 1, it is at this point that one can see a distinct departure from Jacobian methods for handling multiple tasks. In particular, in a priority-based formulation of multiple tasks, subsequent tasks must live in the null-space of the Jacobian of the priority tasks. This implies that convergence guarantees are weakened or lost for low-priority tasks. In contrast, through the output-based CLF framework for tasks, we can simultaneously achieve any number of tasks exactly if the conditions of Theorem 1 are satisfied. ⌟

A. Notions of Lipschitz Continuity for Multivalued Maps

The standard notion of Lipschitz continuity of a function must be extended for the case of multivalued maps. Borrowing definitions from [8], let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively, with $n_1, n_2 > 0$. Given a well-defined norm $||\cdot||$, define the distance between a point $x \in \mathcal{X}$ and a set $S \subset \mathcal{X}$ as $\mathrm{dist}(x, S) = \inf_{x' \in S} ||x - x'||$. A multivalued mapping $F : \mathcal{X} \rightrightarrows \mathcal{Y}$ is *upper Lipschitz continuous (u.L.c)* at a point x_0 if there exists $\delta > 0$ and $L > 0$ (perhaps dependent on x_0) such that for all $x_1 \in B_\delta(x_0) \cap \mathcal{X}$

$$\forall y_1 \in F(x_1), \mathrm{dist}(y_1, F(x_0)) \leq L||x_1 - x_0||. \quad (23)$$

A multivalued mapping $F : \mathcal{X} \rightrightarrows \mathcal{Y}$ is *lower Lipschitz continuous (l.L.c.)* at a point x_0 if there exists $\delta > 0$ and $L > 0$ (perhaps dependent on x_0) such that for all $x_1 \in B_\delta(x_0) \cap \mathcal{X}$

$$\forall y_0 \in F(x_0), \mathrm{dist}(y_0, F(x_1)) \leq L||x_1 - x_0||. \quad (24)$$

Note that if F is a singleton at a point x_0, then the definitions of upper and lower Lipschitz continuity are equivalent. If a mapping is both upper and lower Lipschitz continuous it is *Lipschitz continuous*.

B. Operations on Multivalued Maps

Having established properties of Lipschitz continuity, we can now define operations on multivalued maps and discuss how they can affect the property of Lipschitz continuity. The *intersection* of two multivalued mappings $F_0 : \mathcal{X} \rightrightarrows \mathcal{Y}$ and $F_1 : \mathcal{X} \rightrightarrows \mathcal{Y}$, is itself a multivalued mapping, denoted $(F_0 \cap F_1)$ and defined as $(F_0 \cap F_1)(x) := F_0(x) \cap F_1(x)$. The *composition* of two multivalued mappings $F : \mathcal{X} \rightrightarrows \mathcal{Y}$ and $G : \mathcal{Y} \rightrightarrows \mathcal{Z}$, is a multivalued mapping, defined as $(G \circ F)(x) := \bigcup_{y \in F(x)} G(y)$. The *extension* of a multivalued mapping $F : \mathcal{X} \rightrightarrows \mathcal{Y}$ is a multivalued mapping, defined as $\mathcal{E}F(x) := x \times F(x)$. For multivalued mappings onto the reals, $F : \mathcal{X} \rightrightarrows \mathcal{R}$, the *infimum* will be defined as $\mathcal{I}F(x) := \inf(F(x))$. Note that the extension of an infimum of a multivalued map is given by $\mathcal{EI}F(x) = x \times \mathcal{I}F(x) = x \times \inf(F(x))$.

Proposition 1: Given Lipschitz continuous maps $F_0 : \mathcal{X} \rightrightarrows \mathcal{Y}$, $F_1 : \mathcal{X} \rightrightarrows \mathcal{Y}$, and $G : \mathcal{Y} \rightrightarrows \mathcal{Z}$ then each of the following mappings is Lipschitz continuous: $(F_0 \cap F_1)$, $(G \circ F_0)$, $\mathcal{E}F_0$, and $\mathcal{I}F_0$.

C. Analyzing Quadratic Programming through Operations on Multivalued Maps

To more formally analyze the continuity properties of (20) we will re-define the QP in terms of multivalued maps. Consider the multivalued mappings $C : \mathcal{X} \times \mathbb{R}^{m+p} \rightrightarrows \mathbb{R}$, $S : \mathcal{X} \times \mathbb{R} \rightrightarrows \mathbb{R}^{m+p}$, and $\phi : \mathcal{X} \rightrightarrows \mathbb{R}^{m+p}$ where

$$C(x, u) = u^T H(x) u + 2c(x)^T u \quad (25)$$

$$S(x, v) = \{u \in \mathbb{R}^{m+p} | C(x, u) = v\} \quad (26)$$

$$\phi(x) = \left\{ u \in \mathbb{R}^{m+p} \;\middle|\; \begin{array}{c} A(x)u \leq b(x) \\ A_{eq}(x)u = b_{eq}(x) \end{array} \right\}. \quad (27)$$

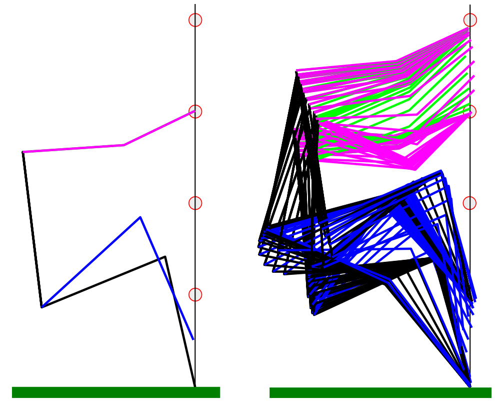

Fig. 1. **Left (a):** Visualization of the robot with the single-domain objective of moving its leg up one rung. **Right (b):** Visualization of the robot performing four distinct reaches (8 total domains) to move up one rung.

We can now define u^* as the set of minimizers given by

$$u^*(x) = S(x, \inf(C(x, \phi(x)))) \cap \phi(x). \quad (28)$$

Or, in an alternative notation let $S^* : \mathcal{X} \times \mathbb{R}^{m+p} \rightrightarrows \mathbb{R}^{m+p}$ be given by

$$S^*(x, u) = \{u' \in \mathbb{R}^{m+p} | C(x, u') = \inf(C(x, u))\} \quad (29)$$

$$= (S \circ \mathcal{EI}C)(x, u) \quad (30)$$

so that $u^*(x) = ((S^* \circ \mathcal{E}\phi) \cap \phi)(x)$.

Proposition 2: If the conditions of Theorem 1 hold at a point x_0, then the mapping S^* will be lower Lipschitz continuous at (x_0, u_0) for any $u_0 \in \mathbb{R}^{m+p}$.

Theorem 2: [Davidson [8], Thm 6.2] *Given data $A \in \mathbb{R}^{n_b \times (m+p)}$, $A_{eq} \in \mathbb{R}^{n_{b_{eq}} \times (m+p)}$, $b \in \mathbb{R}^{n_b}$ and $b_{eq} \in \mathbb{R}^{n_{b_{eq}}}$, let $X(A, A_{eq}, b, b_{eq})$ be the set of extreme points of the convex set ϕ, where*

$$\phi(A, A_{eq}, b, b_{eq}) = \left\{ u \in \mathbb{R}^{m+p} \;\middle|\; \begin{array}{ccc} Au & \leq & b \\ A_{eq}u & = & b_{eq} \end{array} \right\}.$$

If the Mangasarian and Fromovitz (MF) conditions [13] hold, which are

1) *The matrix A_{eq} has full row rank*
2) *There exists a vector u s.t. $Au < b$ and $A_{eq}u = b_{eq}$*

then the extreme points of the feasible set are lower Lipschitz continuous w.r.t. perturbations in (A, A_{eq}, b, b_{eq}). That is, for some $\delta > 0$ there exists $L > 0$ such that for all $(A', A'_{eq}, b', b'_{eq}) \in R_\delta(A, A_{eq}, b, b_{eq})$,

$$||X(A', A'_{eq}, b', b'_{eq}) - X(A, A_{eq}, b, b_{eq})|| \leq L||(A', A'_{eq}, b', b'_{eq}) - (A, A_{eq}, b, b_{eq})||. \quad (31)$$

Proof of the above theorem, along with additional properties of the extreme points of $\phi(x)$ is available in [8].

IV. Simulation Results: Ladder Climbing

In this section we provide a set of simulation examples where multiple control Lyapunov functions as derived in Section II-B are incorporated into a QP-based feedback control law (CLF-QP). Simulation was done in MATLAB, with the QP-based feedback evaluated at each timestep.[1] Although this may be impractical for some systems, software tools such as CVXGEN [14] have greatly reduced the numerical burden of solving a QP. In some cases exact solutions to QPs can be used to avoid the need for an online solver [6].

We have chosen an intentionally simple control task to illustrate the application of Theorem 1. Consider the planar robot shown in Fig. 1 consisting of a single torso link connected to a pair of legs and arms, with two links each, for a total of 9 links. The body coordinates and the stance foot are all assumed to be actuated.[2] In the first three cases we consider the task pictured in Fig. 1(a), when the robot has three contact points with the ladder (two hands and one foot), with the goal of moving the swing foot to the next ladder rung.

Define a vector of generalized coordinates $q \in \mathcal{Q} \subset \mathbb{R}^{11}$ and a vector of torques $\tau \in \mathbb{R}^{11}$. Let $h : \mathcal{Q} \to \mathbb{R}^6$ provide the horizontal and vertical positions of the two hands and stance foot, respectively, as measured from the base of the ladder: $h(q) = (P_{x1}, P_{y1}, P_{x2}, P_{y2}, P_{x3}, P_{y3})^T$. Define a vector of external forces $F = (F_{1x}, F_{1y}, F_{2x}, F_{2y}, F_{3x}, F_{3y})^T$ that act at the three contact points to enforce the holonomic constraints $h(q) = (0, c, 0, c, 0, 0)^T$ for some $c > 0$. The robot's dynamics can now be written in the standard form of (2) with the additional reaction force constraints that $F_{1x}, F_{2x} > 0$ and $F_{3x} < 0$. (The arms can only pull horizontally and the stance foot can only push horizontally.)

The control objective for this task is to zero a pair of output functions $y_1 : \mathcal{Q} \to \mathbb{R}^9$ and $y_2 : \mathcal{Q} \to \mathbb{R}^2$ corresponding to 1) achieving a desired final posture of the robot and 2) tracking a state-based trajectory of the swing foot. Following the constructions of Section II, a pair of corresponding RES-CLFs can be constructed $V_1^\varepsilon : \mathcal{X} \to \mathbb{R}$ and $V_2^\varepsilon : \mathcal{X} \to \mathbb{R}$ for $\mathcal{X} = \mathcal{Q} \times \mathbb{R}^{11}$ that are zeroed by the QP given in (CLF-QP). By varying the value of α used in (CLF-QP) we can affect the continuity properties of the resulting controller.

The first simulation corresponds to the case of $\alpha = 0$. As shown in Fig. 2(a), large "torque chattering" is observed throughout the simulation. Checking the conditions of Theorem 1 we find that $\omega(x) > 0$ for all states encountered, but that $H(x)$ is semidefinite, not strictly positive definite as Theorem 1 requires. In the second case, the choice of $\alpha = 1$ recovers positive definiteness of $H(x)$, but at time $t = 0.088s$ the simulation still fails. See Fig. 2(b). Checking the conditions of Theorem 1 we can confirm that $\omega(x)$ decreases to zero at this time, indicating the control law becomes infeasible. In general, the physical interpretation of $\omega(x)$ is the amount by which the inequality constraints can be made more stringent, but for which the QP will remain solvable. The third case study is illustrated in Fig. 2(c), where the regularization parameter $\alpha = 0.5$ leads to successful completion of the domain. Despite the actuator u_2 remaining saturated for most of the domain the width of the feasible set remains fairly large.

[1]The first three case studies use MATLAB's ode23s for integration, with AbsTol = 1e-6, RelTol = 1e-3, and MaxStep = 1e-4. The last case study uses MATLAB's ode23s for integration, with default tolerances and MaxStep = 1e-2. The quadprog command was used to solve the QP with the active-set algorithm, TolX = TolFun = 1e-10, and max iter = 10,000.

[2]When reaching with one of the legs, the leg remaining in contact with the ladder is considered the stance leg. When reaching with one of the hands, the choice of which leg is the stance leg is arbitrary due to left-right symmetry of the robot.

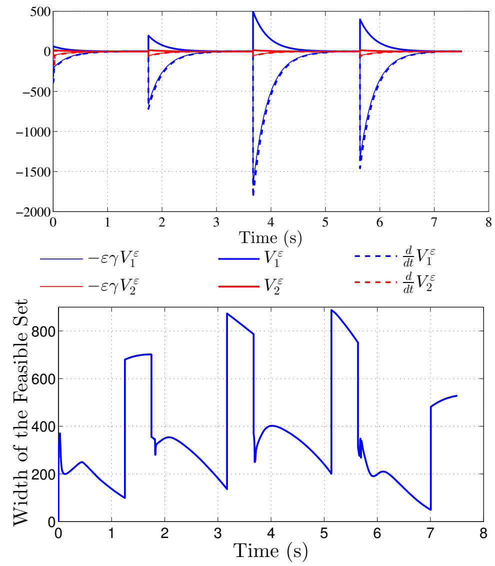

Fig. 3. Exponential convergence of two control Lyapunov functions across the 8 domains that make up climbing a single rung. Control parameters are set at ($\alpha = 0.5, \varepsilon = 5, \tau_{\max} = 500Nm$).

In the final case study we consider the task pictured in Fig. 1(b), with the robot executing multiple domains to climb a single rung of the ladder. The entire sequence consists of four reach phases (both legs followed by both arms). A posture readjustment phase occurs between each pair of subsequent reach phases, where the robot achieves a desired configuration with all four limbs in contact with the ladder. For simplicity a zero velocity reset map is used to transition between phases, where all joint velocities are set to zero when tracking error decreases below a threshold. Figure 3 shows the CLFs rapidly converging within each domain and illustrates how widely the width of the feasible set can vary across the multiple domains of climbing.

V. Conclusions

A QP formulation of multiobjective control was derived for humanoid robots, beginning with Lagrangian dynamics and proceeding through the development of multiple control Lyapunov functions. The main theorem of this paper presents sufficient conditions under which the QP has a unique minimizer that is Lipschitz continuous in the state. This leads to a testable set of conditions for determining whether any set of tasks encoded as CLFs can be simultaneously exponentially stabilized. The theory has been illustrated on the simulation example of a simplified humanoid robot executing a ladder climbing task. In this example we provided cases where the conditions of Theorem 1 are satisfied, and our two control objectives can be simultaneously met. Corollary 1 can then be applied to show that the resulting periodic behavior (i.e. climbing gait) is exponentially stable for a sufficiently rapidly converging RES-CLF.

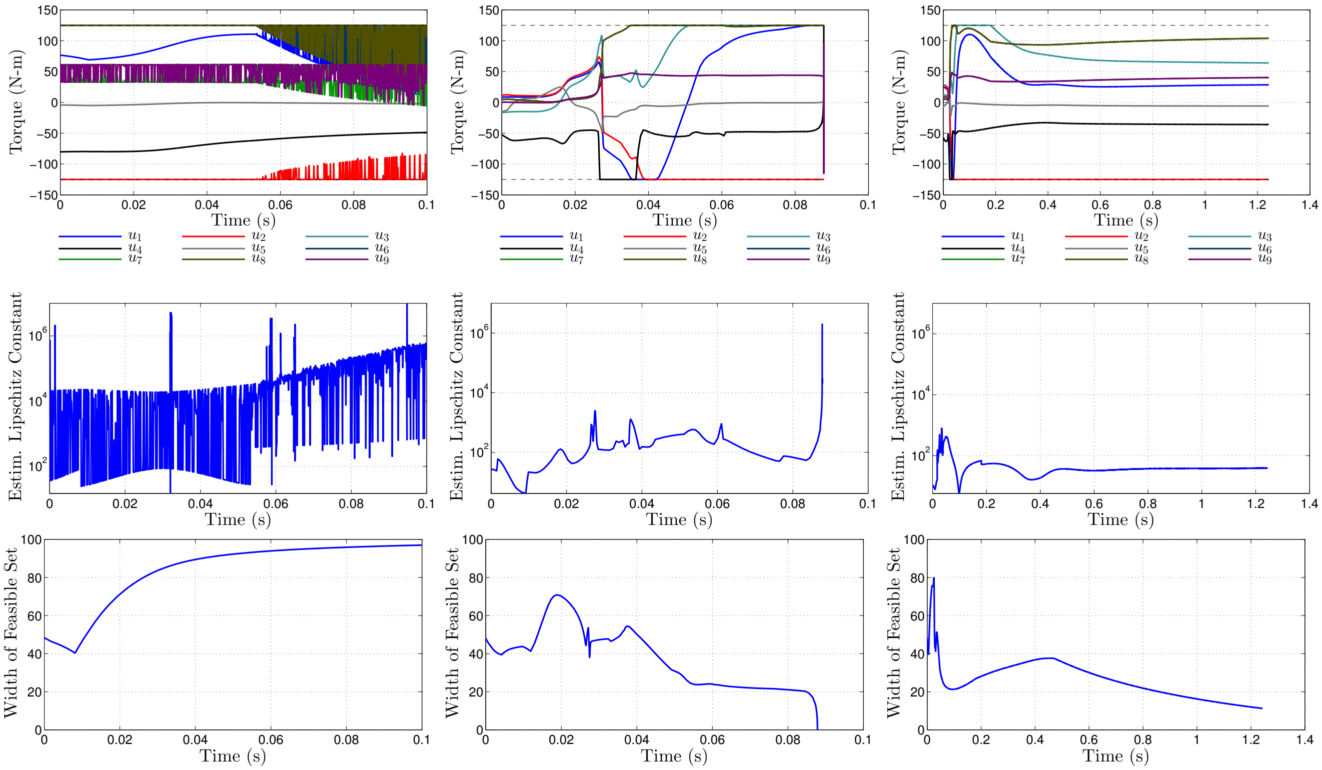

Fig. 2. **Left Column (a):** Simulation of the foot placement task pictured in Fig 1(a), using control parameters ($\alpha = 0, \varepsilon = 10, \tau_{\max} = 125Nm$). This value of α leads to a cost function $H(x)$ is only semidefinite, not strictly positive definite as required by Theorem 1. Rapid chatter in the torques suggest that the underlying control law is not Lipschitz continuous. **Center Column (b):** Using simulation parameters ($\alpha = 1, \varepsilon = 10, \tau_{\max} = 125Nm$) we see that the control law is Lipschitz continuous until t=0.88s when the simulation fails. At this time the width $\omega(x)$ of the feasible set decreases to zero, and the sufficient conditions of Theorem 1 no longer hold. Note that the Lipschitz constant can grow very quickly as ω decreases to 0. **Right Column (c):** Successful simulation of the foot placement task using control parameters ($\alpha = 0.5, \varepsilon = 10, \tau_{\max} = 125Nm$).

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