Characterizing Knee-Bounce in Bipedal Robotic Walking: A Zeno Behavior Approach*

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ABSTRACT

This paper studies the walking behavior of kneed bipedal robots with knee-lock and knee-bounce, formally demonstrating that if knee-locking results in stable bipedal walking, then small amounts of knee-bounce still will result in a walking gait for the robot. To achieve this result, hybrid system models of bipeds are considered wherein knee-bounce corresponds to Zeno behavior. Using results on Zeno stability, we propose a notion of *generalized completion* that allows solutions to be carried beyond the Zeno point, i.e., carried beyond knee-bounce. We assume that the completed hybrid system has a periodic orbit when the impacts are perfectly plastic—a *plastic periodic orbit*, or walking gait with knee-lock. The main result of this paper is that when the assumption of perfectly plastic impacts is relaxed, if the plastic periodic orbit is stable and the Zeno point is Zeno stable, then there exists a periodic orbit in the case of non-plastic impacts, i.e., a *Zeno periodic orbit* corresponding to walking with knee-bounce. This formal result is applied to a specific example of a bipedal robot with knees.

Categories and Subject Descriptors

G.1.0 [**Numerical Analysis**]: General—*Stability (and instability)*; I.6.8 [**Simulation and Modeling**]: Types of Simulation—*Continuous, Discrete Event*

General Terms

Theory

Keywords

Hybrid mechanical systems, Bipedal robotic walking, Zeno behavior, Stability, Periodic orbits.

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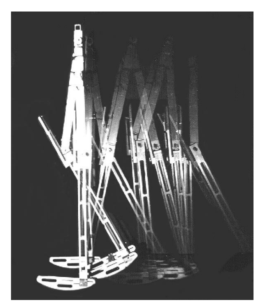
1. INTRODUCTION

Mechanical knees are an important component of achieving natural and "human-like" walking in bipedal robots [6, 17]. Mechanical "knee-caps", i.e., mechanical stops, are typically added to these mechanical knees to prevent the leg from hyper-extending (see the figure on the right for a bipedal robot with mechanical knees that lock built by Cornell[1], replicating the passive biped with knees of McGeer [1, 17]). Yet with this benefit comes a cost: *knee-bounce*, which occurs when the shin bounces off the mechanical stop repeatedly as the leg attempts to lock. It has been shown that this behavior can destabilize the robot in certain situations; see [1] for a passive bipedal robot that falls due to knee-bounce. To address this problem, mechanical catches are often added to the knees to prevent bouncing behavior, i.e., *knee-lock* is enforced through mechanical means. It can be seen in [1] that adding knee-lock to the robot resulted in stable walking. Although the addition of knee-lock to a robot can result in walking where it would not be present with knee-bounce, it comes at the cost of additional mechanical complexity.

This motivates the question: *is it necessary to add mechanical catches that enforce knee-lock to bipedal robots to obtain walking?* This paper formally shows that mechanical catches are not always necessary to achieve bipedal walking, i.e., that if stable walking exists for a robot with knee-lock, then walking will also exist with knee-bounce as long as the knee-bounce is sufficiently small. To achieve this result, it is necessary to understand what knee-lock and knee-bounce correspond to in a formal setting.



Bipedal robots are naturally modeled by systems with discrete and continuous behavior: hybrid systems. The continuous component consists of dynamics dictated by Lagrangians modeling the robot with the number of contact points (such as foot and knee contact) enforced through holonomic constraints. The discrete behavior occurs when the number of contact points changes, e.g., the knee locks or unlocks or the foot strikes the ground, resulting in an instantaneous change in the velocity of the system. In the setting of hybrid systems, knee-bounce and knee-lock can be formally understood to be, respectively, a result of non-plastic

[1]Photo credit: Rudra Pratap, http://ruina.tam.cornell.edu/hplab/pdw.html

and plastic impacts at the knee. In this setting, knee-bounce corresponds to *Zeno behavior*—when an infinite number of discrete transitions, or impacts, occur in a finite amount of time—where the point to which the Zeno solution converges, the *Zeno point*, corresponds to the leg being straight. The behavior of knee-bounce can thus be compared to the behavior of knee-lock, where the Zeno point is reached instantaneously due to the plastic impact.

Using the formalisms of hybrid systems and Zeno behavior, we can approach the problem of knee-bounce rigorously. In particular, we consider *Lagrangian hybrid systems* which model mechanical systems undergoing impacts as dictated by *unilateral constraints* on the configuration space; the amount of energy lost through impact is dictated by the *coefficient of restitution.* Hybrid systems of this form have a single discrete domain and can display Zeno behavior. Easily verifiable conditions on the existence of this behavior based upon the coefficient of restitution and the unilateral constraint function have been proven [13, 14, 15]. The first result of this paper is a bound on the distance between the *Zeno point*—the limit point of a Zeno solution—for plastic and non-plastic impacts. This result is essential in establishing the main result of this paper.

When there exists Zeno behavior, the hybrid system can be *completed* to allow for solutions to extend beyond the finite Zeno time. In particular, this paper presents a notion of *generalized completion*, extending previous notions of hybrid system completion [3, 4, 18, 22, 23] to a setting that will be applicable to bipedal robot models. Specifically, an additional domain is added to the hybrid system—the *post-Zeno* domain—which enforces the unilateral constraint as a holonomic constraint. A solution transitions to this domain when the Zeno point is reached, and transitions back when conditions in the post-Zeno domain are reached (generalizing the traditional transition back to the pre-Zeno domain based upon the Lagrange multipliers associated to the holonomic constraint). For a bipedal robot, the pre-Zeno domain is when the leg is bent, the post-Zeno domain is when the leg is straight, a transition to the post-Zeno domain occurs when the knee locks (in finite time for knee-bounce and instantaneously for knee-lock), and a transition to the pre-Zeno domain occurs when the foot strikes the ground.

The objective of this paper is to consider periodic orbits in completed hybrid systems and to show that the existence of periodic orbits in the case of plastic impacts—termed *plastic* periodic orbits—implies the existence of periodic orbits for non-plastic impacts—termed *Zeno* periodic orbits. Beginning with a plastic periodic orbit, the assumption of plastic impacts is relaxed, i.e., the coefficient of restitution is no longer assumed to be zero. In this case, the hybrid system will display non-trivial Zeno behavior. The main result is that if the plastic periodic orbit is stable and the Zeno point is Zeno stable, then for sufficiently small coefficients of restitution there exists a Zeno periodic orbit. In order to demonstrate the practical usefulness of the results of this paper, we apply them to a bipedal robot model with knees. In this setting plastic periodic orbits correspond to knee-lock and Zeno periodic orbits correspond to knee-bounce. We show numerically that the conditions of the main result are satisfied, and we confirm through simulation that the existence of a stable walking gait with knee-lock implies the existence of a stable walking gait with knee-bounce.

This paper, therefore, provides the first steps towards understanding Zeno behavior–and more general phenomena unique to hybrid systems—in complex hybrid mechanical systems. Due to the relationship between knee-bounce in robotic walkers and Zeno behavior shown in this paper, it is evident that understanding the abstract formalisms used to model physical mechanical systems can lead to important insights into the behavior of these systems; moreover, these insights can be used to aid in the design of these systems through the knowledge that design decisions can affect the behavior of the system. For example, the results of this paper imply that mechanical knee-locks are not necessarily when constructing physical bipedal robots as long as the knee bounce (or coefficient of restitution) is kept small.

2. HYBRID MECHANICAL SYSTEMS

Bipedal walkers are naturally modeled by hybrid systems. This section, therefore, introduces the basic terminology of hybrid systems in a general enough setting so as to formally describe both bipedal robotic models and completed hybrid systems.

Definition 1. *A* hybrid system *is a tuple*

$$\mathscr{H} = (\Gamma, D, G, R, F),$$

where

- $\Gamma = (V, E)$ *is an oriented graph, i.e.,* V *and* E *are a set of vertices and edges, respectively, and there exists a source function* $\mathrm{sor} : E \to V$ *and a target function* $\mathrm{tar} : E \to V$ *which associates to an edge its source and target, respectively.*
- $D = \{D_v\}_{v \in V}$ *is a set of domains, where* $D_v \subseteq \mathbb{R}^{n_v}$ *is a smooth submanifold of* $\mathbb{R}^{n_v}$*,*
- $G = \{G_e\}_{e \in E}$ *is a set of guards, where* $G_e \subseteq D_{\mathrm{sor}(e)}$*,*
- $R = \{R_e\}_{e \in E}$ *is a set of reset maps, where* $R_e : G_e \to D_{\mathrm{tar}(e)}$ *is a smooth map,*
- $F = \{f_v\}_{v \in E}$*, where* f_v *is a smooth dynamical system on* D_v*, i.e.,* $\dot{x} = f_v(x)$ *for* $x \in D_v$*.*

Definition 2. *An execution of a hybrid system* $\mathscr{H}$ *is a tuple* $\chi = (\Lambda, \mathcal{I}, \rho, \mathcal{C})$*, where*

- $\Lambda = \{0, 1, 2, \ldots\} \subseteq \mathbb{N}$ *is a finite or infinite indexing set,*
- $\mathcal{I} = \{I_i\}_{i \in \Lambda}$ *where for each* $i \in \Lambda$*,* I_i *is defined as follows:* $I_i = [t_i, t_{i+1}]$ *if* $i, i+1 \in \Lambda$ *and* $I_{N-1} = [t_{N-1}, t_N]$ *or* $[t_{N-1}, t_N)$ *or* $[t_{N-1}, \infty)$ *if* $|\Lambda| = N$*,* N *finite. Here, for all* $i, i+1 \in \Lambda$*,* $t_i \leq t_{i+1}$ *with* $t_i, t_{i+1} \in \mathbb{R}$*, and* $t_{N-1} \leq t_N$ *with* $t_{N-1}, t_N \in \mathbb{R}$*,*
- $\rho : \Lambda \to V$ *is a map such that for all* $i, i+1 \in \Lambda$*,* $(\rho(i), \rho(i+1)) \in E$*. This is the discrete component of the execution,*
- $\mathcal{C} = \{c_i\}_{i \in \Lambda}$ *is a set of continuous trajectories, and they must satisfy* $\dot{c}_i(t) = f_{\rho(i)}(c_i(t))$ *for* $t \in I_i$*.*

We require that when i*,* $i+1 \in \Lambda$*,*

$$\begin{array}{ll} \text{(i)} & c_i(t) \in D_{\rho(i)} \ \forall \ t \in I_i \\ \text{(ii)} & c_i(t_{i+1}) \in G_{(\rho(i), \rho(i+1))} \\ \text{(iii)} & R_{(\rho(i), \rho(i+1))}(c_i(t_{i+1})) = c_{i+1}(t_{i+1}). \end{array} \tag{1}$$

When $i = |\Lambda| - 1$*, we still require that (i) holds. The initial condition for the hybrid execution is* $c_0(t_0) \in D_{\rho(0)}$*.*

Simple hybrid systems. A simple hybrid system is a hybrid system with a single domain and edge. Systems of this form have been widely studied, especially with respect to Zeno behavior [3, 14, 22]. In addition, the model with non-plastic impacts to be considered later will be represented by a simple hybrid system.

Formally, a *simple hybrid system* is a hybrid system with $\Gamma = (\{v\}, \{e = (v, v)\})$. Since in this case there is only a single domain, guard, reset map and vector field, we write a simple hybrid system as a tuple:

$$\mathscr{SH} = (D, G, R, f),$$

where D is a domain (not a set of domains), G is a guard, R is a reset map and f is a vector field.

Consider an execution of a simple hybrid system $\chi^{\mathscr{SH}_{\mathbf{L}}} = (\Lambda, \mathcal{I}, \rho, \mathcal{C})$. Since there is only one domain, the only choice for the discrete component of the execution is $\rho(i) \equiv v$ Therefore, we can write an execution of a simple hybrid system as $\chi^{\mathscr{SH}_{\mathbf{L}}} = (\Lambda, \mathcal{I}, \mathcal{C})$.

We now consider simple hybrid systems modeling mechanical systems: *Lagrangian hybrid systems.* These systems are obtained from *hybrid Lagrangians* which consist of a configuration space, a Lagrangian and a unilateral constraint (systems of this form have been well-studied in the mechanics literature [4, 5, 18, 19, 27]).

Dynamical systems from Lagrangians. Let $q \in Q$ be the *configuration* of a mechanical system.[2] In this paper, we will consider Lagrangians, $L : TQ \to \mathbb{R}$, describing mechanical or robotic systems, which are of the form

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q), \tag{2}$$

with $M(q)$ the (positive definite) inertial matrix, $\frac{1}{2}\dot{q}^T M(q)\dot{q}$ the kinetic energy and $V(q)$ the potential energy. Assume there is a feedback *control law* $\Upsilon(q, \dot{q})$, which is a given smooth function $\Upsilon : TQ \to Q$. In this case, the Euler-Lagrange equations yield the (controlled) equations of motion for the system given in coordinates by:

$$M(q)\ddot{q} + C(q, \dot{q}) + N(q) = \Upsilon(q, \dot{q}), \tag{3}$$

where $C(q, \dot{q})$ is the vector of centripetal and Coriolis terms (cf. [21]) and $N(q) = \frac{\partial V}{\partial q}(q)$. Defining the *state* of the system as $x = (q, \dot{q})$, the Lagrangian vector field, f_L, associated to L takes the familiar form:

$$\begin{aligned} \dot{x} &= f_L(x) \\ &= \begin{pmatrix} \dot{q} \\ M(q)^{-1}(-C(q, \dot{q}) - N(q) + \Upsilon(q, \dot{q})) \end{pmatrix}. \end{aligned} \tag{4}$$

Holonomic constraints. We now define the holonomically constrained dynamical system with a Lagrangian L and a *holonomic constraint* $\eta : Q \to \mathbb{R}$. For such systems, the constrained equations of motion can be obtained from the equations of motion for the unconstrained system (3), and are given by (cf. [21])

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q) = d\eta(q)^T \lambda + \Upsilon(q, \dot{q}), \tag{5}$$

where λ is the Lagrange multiplier which represents the contact force and $d\eta(q) = \left(\frac{\partial \eta}{\partial q}(q)\right)^T$.

[2]For simplicity, in the models considered, we assume that the configuration space is identical to $\mathbb{R}^n$

Differentiating the constraint equation $\eta(q) = 0$ twice with respect to time and substituting the solution for $\ddot{q}$ in (5), the solution for the constraint force λ is obtained (see [21]). From the constrained equations of motion (5) and (3), for $x = (q, \dot{q})$, we get the vector field

$$\dot{x} = f_L^{\eta}(x) = f_L(x) + \begin{pmatrix} 0 \\ M(q)^{-1} d\eta(q)^T \lambda(q, \dot{q}) \end{pmatrix}. \tag{6}$$

Unilateral Constraints. The domain, guard and reset map (for knee lock) will be obtained from *unilateral constraint* $h : Q \to \mathbb{R}$ which gives the set of admissible configurations of the system; we assume that the zero level set $h^{-1}(0)$ is a smooth manifold.

Define the domain and guard, respectively, as

$$\begin{aligned} D_h &= \{(q, \dot{q}) \in TQ : h(q) \geq 0\}, \\ G_h &= \{(q, \dot{q}) \in TQ : h(q) = 0 \text{ and } dh(q)\dot{q} \leq 0\}. \end{aligned} \tag{7}$$

The reset map associated to a unilateral constraint is obtained through impact equations of the form (see [5, 19]):

$$R_h(q, \dot{q}) = \begin{pmatrix} q \\ \dot{q} - (1 + \varepsilon)\frac{dh(q)\dot{q}}{dh(q)M(q)^{-1}dh(q)^T} M(q)^{-1} dh(q)^T \end{pmatrix}. \tag{8}$$

Here $0 \leq \varepsilon \leq 1$ is the *coefficient of restitution*, which is a measure of the energy dissipated through impact; for a perfectly plastic impact $\varepsilon = 0$ and for a perfectly elastic impact $\varepsilon = 1$. This reset map corresponds to rigid-body collision under the assumption of *frictionless impact.* Examples of more complicated collision laws that account for friction can be found in [5] and [27].

Definition 3. *A simple hybrid Lagrangian is defined to be a tuple* $\mathbf{L} = (Q, L, h)$*, where*

- Q *is the configuration space (assumed to be* $\mathbb{R}^n$*),*
- $L : TQ \to \mathbb{R}$ *is a Lagrangian of the form* (2)*,*
- $h : Q \to \mathbb{R}$ *is a unilateral constraint.*

Simple Lagrangian hybrid systems. For a given Lagrangian, there is an associated dynamical system. Similarly, given a hybrid Lagrangian $\mathbf{L} = (Q, L, h)$ the *simple Lagrangian hybrid system (SLHS)* $\mathscr{SH}_{\mathbf{L}}$, associated to $\mathbf{L}$ is the simple hybrid system: $\mathscr{SH}_{\mathbf{L}} = (D_h, G_h, R_h, f_L)$.

Remark 1. *We often will want to make clear the dependence of the reset map on the coefficient of restitution* ε*, in which case we will write* R_h^{ε}*. Therefore, in the case of perfectly plastic impacts, the reset map is given by* R_h^0*. In the case of SLHS's, we will use the same convention writing* $\mathscr{SH}_{\mathbf{L}}^{\varepsilon}$*.*

3. ZENO BEHAVIOR

We now introduce Zeno behavior and the corresponding notion of Zeno equilibria, and we consider the stability of these equilibria. The first result of the paper is then presented; this gives bounds in the distance between Zeno points for plastic and non-plastic impacts—a result that is vital in proving the main result of this paper.

Definition 4. *An execution $\chi^{\mathscr{H}}$ is Zeno if $\Lambda = \mathbb{N}$ and*

$$t_\infty := \lim_{i\to\infty} t_i - t_0 = \sum_{i=0}^{\infty} t_{i+1} - t_i < \infty.$$

Here t_∞ is called the Zeno time.

Zeno behavior in SLHS's. If $\chi^{\mathscr{SH}_{\mathbf{L}}}$ is a Zeno execution of a SLHS $\mathscr{SH}_{\mathbf{L}}$, then its *Zeno point* is defined to be

$$x_\infty = (q_\infty, \dot{q}_\infty) = \lim_{i\to\infty} c_i(t_i) = \lim_{i\to\infty} (q_i(t_i), \dot{q}_i(t_i)). \qquad (9)$$

These limit points are intricately related to a type of equilibrium point that is unique to hybrid systems: Zeno equilibria.

Definition 5. *A Zeno equilibrium point of a simple hybrid system $\mathscr{SH}$ is a point $x^* \in G$ such that $R(x^*) = x^*$ and $f(x^*) \neq 0$.*

We also can consider the stability of Zeno equilibria (see [7, 8] for complementary notions of stability as it relates to Zeno behavior).

Definition 6. *A Zeno equilibrium x^* of a simple hybrid system $\mathscr{SH}$ is bounded-time locally Zeno stable if for every $U \subset D$ and every $\epsilon > 0$, there exists an open set $W \subset U$ with $x^* \in W$ such that for every $x_0 \in W$, there exists a unique*[3] *execution χ with $c_0(t_0) = x_0$ and $\Lambda = \mathbb{N}$. This execution is Zeno with $t_\infty < \epsilon$ and $c_i(t) \in U$ for all $i \in \mathbb{N}$ and $t \in I_i$.*

Zeno stability in simple Lagrangian hybrid systems. If $\mathscr{SH}_{\mathbf{L}}$ is a SLHS, then due to the special form of these systems we find that the point $(q^*, \dot{q}^*)$ is a Zeno equilibria iff $\dot{q}^* = P_h(q, \dot{q}^*)$, with P_h given in (8). In particular, the special form of P_h implies that this holds iff $dh(q^*)\dot{q}^* = 0$. Therefore the set of all Zeno equilibria for a SLHS is:

$$\mathcal{Z} = \{(q, \dot{q}) \in D_h : h(q) = 0 \text{ and } dh(q)\dot{q} = 0\}. \qquad (10)$$

Note that if $\dim(Q) > 1$, the Zeno equilibria in Lagrangian hybrid systems are always non-isolated.

Let $\ddot{h}(q, \dot{q})$ be the acceleration of $h(q)$ along trajectories of the unconstrained dynamics (3), given by:

$$\ddot{h}(q, \dot{q}) = \dot{q}^T H(q)\dot{q} + dh(q)M(q)^{-1}(-C(q, \dot{q})\dot{q} - N(q)),$$

where $H(q)$ is the Hessian of h at q. The following theorem, which was presented in [13, 14, 15], provides sufficient conditions for existence of Zeno executions in the vicinity of a Zeno equilibrium point.

Theorem 1 **([13, 15]).** *Let $\mathscr{SH}_{\mathbf{L}}$ be a simple Lagrangian hybrid system and let $x^* = (q^*, \dot{q}^*)$ be a Zeno equilibrium point of $\mathscr{SH}_{\mathbf{L}}$. If $0 \leq \varepsilon < 1$ and $\ddot{h}(q^*, \dot{q}^*) < 0$, then $(q^*, \dot{q}^*)$ is bounded-time locally Zeno stable.*

Note that Thm. 1 was proven in [13, 15] using “Lyapunov-like” functions which mapped hybrid systems to *first quadrant interval hybrid systems* which can be viewed as the “simplest Zeno stable systems.” In particular, in the case of mechanical systems, the following function was considered:

$$\psi(q, \dot{q}) = \begin{pmatrix} \dot{h}(q, \dot{q}) + \sqrt{\dot{h}(q, \dot{q})^2 + 2h(q)} \\ -\dot{h}(q, \dot{q}) + \sqrt{\dot{h}(q, \dot{q})^2 + 2h(q)} \end{pmatrix},$$

which has the following properties (as proven in [15]) for a Zeno execution χ with initial condition $c_0(t_0) = x_0 \in G_h$:

(P1) From the definition of the reset map R_h^ε

$$\psi(c_i(t_i))_1 = \varepsilon \psi(c_{i-1}(t_i))_2.$$

(P2) From the proof of Thm. 2 in [15] (from combining equations (4) and (5) and using (P1)), the Zeno time has an upper bound:

$$t_\infty < \frac{\varepsilon}{1-\varepsilon} \psi(x_0)_2 \frac{1}{|\alpha|}$$

where $\alpha = \max_{x\in U} d\psi(x)_1 f_L(x)$.

(P3) Again from the proof of Thm. 2 in [15] (specifically by combining equations (5) and (6) with (P1))

$$\sum_{i=1}^{\infty} \psi(c_i(t_{i+1}))_2 \leq \frac{\varepsilon}{1-\varepsilon} \psi(x_0)_2 \left|\frac{\beta}{\alpha}\right|$$

where $\beta = \max_{x\in U} d\psi(x)_2 f_L(x)$.

These properties will be utilized in proving the first result of this paper, but first some set-up is necessary.

Zeno points for plastic and non-plastic impacts. Let $\mathscr{SH}_{\mathbf{L}}^\varepsilon$ be a SLHS with a coefficient of restitution $\varepsilon > 0$. For $x_0 \in G_h$, if there is a Zeno execution χ with this point as its initial condition then it has a well-defined Zeno point $x_\infty^\varepsilon(x_0)$. We are interested in comparing this Zeno point with the point obtained by applying a perfectly plastic impact $x_\infty^0(x_0) = R_h^0(x_0)$, which is just the Zeno point of an execution of $\mathscr{SH}_{\mathbf{L}}^0$ with initial condition x_0. Necessarily, it follows that $x_\infty^0(x_0)$ is a Zeno equilibrium point.

We are interested in comparing $x_\infty^\varepsilon(x_0)$ and $x_\infty^0(x_0) = R_h^0(x_0)$. Intuitively, these two points should converge toward one another in a continuous fashion as $\varepsilon \to 0$. This is what the following proposition verifies.

Proposition 1. *Let $\mathscr{SH}_{\mathbf{L}}^\varepsilon$ be a simple Lagrangian hybrid system with a coefficient of restitution $\varepsilon > 0$. Let $x^* = (q^*, \dot{q}^*)$ be a Zeno equilibrium point of $\mathscr{SH}_{\mathbf{L}}^\varepsilon$ that is bounded-time locally Zeno stable, and let W, U and ϵ be as in Def. 6. If $x_0 = (q_0, \dot{q}_0) \in G_h \cap W$, then there exist positive constants A_1, A_2 and A_3 such that:*

$$\begin{aligned} &||x_\infty^\varepsilon(x_0) - x_\infty^0(x_0)|| \\ &\quad < \varepsilon \left(A_1 + \frac{1}{1-\varepsilon} A_2 + \frac{1+\varepsilon}{1-\varepsilon} A_3 \right) |dh(q_0)\dot{q}_0|. \end{aligned} \qquad (11)$$

Note that the constants A_1, A_2 and A_3 in this lemma simply give bounds on the growth of the vector field and reset map over the region U. As a result, this lemma has a clear physical intuition. Basically the distance between the Zeno point for a plastic and non-plastic impact is determined by two main factors: the coefficient of restitution and the speed when the guard is reached, $|\dot{h}(q_0, \dot{q}_0)| = |dh(q_0)\dot{q}_0|$.

Proof. Since $x_0 \in G_h$, it follows that $t_0 = t_1$ and $c_0(t_1) = c_0(t_0) = x_0$ and so by the definition of R_h^ε and ψ

$$||c_1(t_1) - x_\infty^0(x_0)|| = ||R_h^\varepsilon(x_0) - R_h^0(x_0)|| \leq \varepsilon K \psi(x_0)_2$$

where

$$K = \max_{x=(q,\dot{q})\in U} \frac{1}{2} \frac{||M(q)^{-1} dh(q)^T||}{dh(q)M(q)^{-1}dh(q)^T}.$$

[3]This is just the *maximal* execution with initial condition $c_0(t_0) = x_0$.

More generally, for $x \in G_h$, again from the definition of R_h^ε and ψ,

$$||R_h(x) - x_\infty^0(x_0)|| \leq ||x - x_\infty^0(x_0)|| + (1+\varepsilon)K\psi(x)_2.$$

Therefore, for $i \geq 2$,

$$\begin{aligned}&||c_i(t_i) - x_\infty^0(x_0)|| \leq \\ &\qquad ||c_{i-1}(t_i) - x_\infty^0(x_0)|| + (1+\varepsilon)K\psi(c_{i-1}(t_i))_2\end{aligned}$$

and

$$\begin{aligned}&||c_{i-1}(t_i) - x_\infty^0(x_0)|| \\ &\quad = \quad ||c_{i-1}(t_{i-1}) - x_\infty^0(x_0) + \int_{t_{i-1}}^{t_i} f_L(x(\tau))d\tau|| \\ &\quad \leq \quad ||c_{i-1}(t_{i-1}) - x_\infty^0(x_0)|| + F(t_i - t_{i-1})\end{aligned}$$

where $F = \max_{x\in U} ||f_L(x)||$. It follows that

$$\begin{aligned}||c_i(t_i) - x_\infty^0(x_0)|| &\leq ||c_{i-1}(t_{i-1}) - x_\infty^0(x_0)|| \\ &\quad + F(t_i - t_{i-1}) + (1+\varepsilon)K\psi(c_{i-1}(t_i))_2.\end{aligned}$$

By induction, or through simple iteration, we thus have that

$$\begin{aligned}&||c_i(t_i) - x_\infty^0(x_0)|| \leq \varepsilon K\psi(x_0)_2 \\ &\qquad + F\sum_{j=1}^{i-1}(t_{j+1} - t_j) + (1+\varepsilon)K\sum_{j=1}^{i-1}\psi(c_j(t_{j+1}))_2.\end{aligned}$$

It follows from (P1), (P2) and (P3), together with the fact that $t_0 = t_1$, that

$$\begin{aligned}&||x_\infty^\varepsilon(x_0) - x_\infty^0(x_0)|| = \lim_{i\to\infty} ||c_i(t_i) - x_\infty^0(x_0)|| \\ &\quad \leq \varepsilon K\psi(x_0)_2 \\ &\qquad + F\sum_{j=1}^{\infty}(t_{j+1} - t_j) + (1+\varepsilon)K\sum_{j=1}^{\infty}\psi(c_j(t_{j+1}))_2 \\ &\quad \leq \varepsilon K\psi(x_0)_2 \\ &\qquad + F\frac{\varepsilon}{1-\varepsilon}\psi(x_0)_2\frac{1}{|\alpha|} + (1+\varepsilon)K\frac{\varepsilon}{1-\varepsilon}\psi(x_0)_2\left|\frac{\beta}{\alpha}\right|\end{aligned}$$

Since $\psi(x_0)_2 = 2|dh(q_0)\dot{q}_0|$ by definition, picking

$$A_1 = 2K, \qquad A_2 = 2F\frac{1}{|\alpha|}, \qquad A_3 = 2K\left|\frac{\beta}{\alpha}\right|$$

gives the desired result. □

4. COMPLETED HYBRID SYSTEMS & ZENO PERIODIC ORBITS

Completed hybrid systems allow Zeno executions to be carried past the Zeno point. Let $\mathscr{SH}_\mathbf{L}$ be a SLHS then, as the execution converges toward the Zeno point, $h \to 0$. This implies that after the Zeno point is reached, there should be a switch to a holonomically constrained dynamical system with holonomic constraint $\eta = h$. Let $\mathscr{D}_h = (\mathcal{Z}, f_L^h)$ be the dynamical system obtained from this unilateral constraint as in (5) with $\mathcal{Z}$ the set in (10).

Traditionally, completed hybrid systems have been defined in the following manner [3, 4, 18, 22, 23, 24] (and are often termed complementary Lagrangian hybrid systems): if $\mathbf{L}$ is a simple hybrid Lagrangian and $\mathscr{SH}_\mathbf{L}$ the corresponding

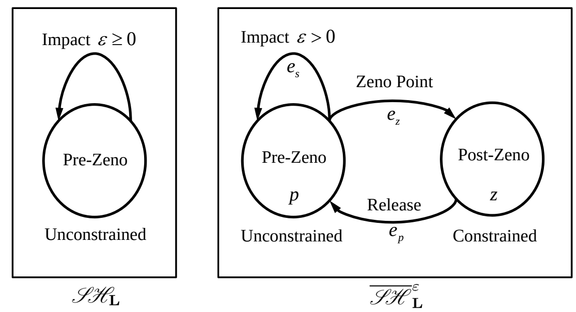


Figure 1: A graphical representation of a SHS and its associated completed hybrid system.

SLHS, the corresponding *completed Lagrangian hybrid system*[4] is:

$$\overline{\mathscr{SH}}_\mathbf{L} := \begin{cases} \mathscr{D}_h & \text{if} \quad h(q) = 0\ ,\ dh(q)\dot{q} = 0, \\ & \qquad \text{and } \lambda(q,\dot{q}) > 0 \\ \mathscr{SH}_\mathbf{L} & \qquad \text{otherwise} \end{cases}$$

where λ is the Lagrange multiplier obtained from h. Systems of this form have been well-studied in the above references, and conditions have been given on how to practically simulate completed hybrid systems by truncating the executions in a formal manner (see [23, 24]).

While the notion of a completed hybrid system has proven very useful, we wish to extend it to include the possibility of unilateral constraints in the "post-Zeno" domain $\mathscr{D}_h$ that would cause the lagrange multiplier to switch sign (causing a switch back to the pre-Zeno domain). The second consideration is that we want this more general definition of a completed hybrid system to include as a special case the previous definition while simultaneously being able to model physical situations that occur with bipeds.

With this in mind, we consider the following definition of a *generalized completed hybrid system* (see Fig. 1).

Definition 7. *Let $\mathscr{SH}_\mathbf{L}$ be a SLHS associated to a hybrid Lagrangian $\mathbf{L} = (Q, L, h)$. A completed SLHS*[5] *is a tuple: $\overline{\mathscr{SH}}_\mathbf{L}^\varepsilon := (\overline{\Gamma}, \overline{D}, \overline{G}, \overline{R}, \overline{F})$, where*

- $\overline{\Gamma} = \{(p,z), e_s = (p,p), e_z = (p,z), e_p = (z,p)\}$,
- $\overline{D} = \{\overline{D}_p, \overline{D}_z\}$ *where* $\overline{D}_p = D_h$ *and* $\overline{D}_z \subset \mathcal{Z}$ *satisfying:*
$$\lambda(q,\dot{q}) \geq 0 \ \text{ if } \ (q,\dot{q}) \in \overline{D}_z, \qquad (12)$$
- $\overline{G} = \{\overline{G}_{e_s}, \overline{G}_{e_z}, \overline{G}_{e_p}\}$ *where* $\overline{G}_{e_s} = G_h \backslash \mathcal{Z}$, $\overline{G}_{e_z} = \mathcal{Z}$ *and* $\overline{G}_{e_p} \subset \overline{D}_z$,
- $\overline{R} = \{\overline{R}_{e_s}, \overline{R}_{e_z}, \overline{R}_{e_p}\}$ *where* $\overline{R}_{e_s} = R_h$ *(which depends on the coefficient of restitution* ε*),* $\overline{R}_{e_z} = I$ *and* $\overline{R}_{e_p} : \overline{G}_{e_p} \to D_p$ *satisfying:*
$$\lambda(\overline{R}_{e_p}(q,\dot{q})) \leq 0 \ \text{ for } \ (q,\dot{q}) \in \overline{G}_{e_p}, \qquad (13)$$
- $\overline{F} = \{\overline{f}_p, \overline{f}_z\}$ *where* $\overline{f}_p = f_L$ *and* $\overline{f}_z = f_L^h$.

[4]As was originally pointed out in [3], this terminology (and notation) is borrowed from topology, where a metric space can be completed to ensure that "limits exist."

[5]We make the dependence of the system on the coefficient of restitution ε explicit since it is the main object of interest.

Note that the guard that forces a transition from the "pre-Zeno" domain $\overline{D}_p$ to the "post-Zeno" domain $\overline{D}_z$ is just the set of Zeno equilibria for the simple hybrid system $\mathscr{SH}_{\mathbf{L}}$, i.e., a transition only occurs at the limit point of a Zeno execution. Also note that conditions (12) and (13) are consistency conditions that ensure that the constraint force has the right sign through the "post-Zeno" domain, and that when a transition back to the "pre-Zeno" domain occurs the constraint will no longer be enforced. Finally note that the traditional notion of completion is just a special case of Def. 7 with:

$$\overline{D}_z = \{(q, \dot{q}) \in \mathcal{Z} : \lambda(q, \dot{q}) \geq 0\},$$

(the largest domain satisfying (12)),

$$\overline{G}_{e_p} = \{(q, \dot{q}) \in \overline{D}_z : \lambda(q, \dot{q}) = 0\},$$

and $\overline{R}_{e_p} = I$ (which therefore satisfies (13)).

To better understand the motivation for the definition of a generalized completed hybrid system, and why the traditional definition must be extended, the specific application of bipedal robots must be considered. In this case, the traditional case where the solution switches back to the post-Zeno domain with an identity reset map would not be consistent with the bipedal robotic model. In particular, in the case of a biped the reset map $\overline{R}_{e_p}$ is the reset map associated with foot impact (see Fig. 4 for a graphical representation). The specifics of how a generalized completed hybrid system is obtained in bipedal walking will be presented in detail in Sec. 6, but before studying these systems in the context of walking, their properties must be studied.

Zeno points in completed systems. If $\overline{\mathscr{SH}}^{\varepsilon}_{\mathbf{L}}$ is a completed SLHS, then for executions with initial conditions in the pre-Zeno domain we again can consider the Zeno point of these executions in the case when the coefficient of restitution $\varepsilon > 0$. Specifically, let $\overline{\chi} = (\Lambda, \mathcal{I}, \rho, \mathcal{C})$ be an execution of $\overline{\mathscr{SH}}^{\varepsilon}_{\mathbf{L}}$ with $c_0(t_0) \in \overline{D}_p$. By the definition of the completed system and because $\varepsilon > 0$, the solution is Zeno and will never leave the pre-Zeno domain $\overline{D}_p$. Therefore, for this execution $\rho(i) \equiv p$. The Zeno point is thus given in (9) as in the case of simple hybrid systems.

In the case when $\varepsilon = 0$, the completed system $\overline{\mathscr{SH}}^{0}_{\mathbf{L}}$ is "instantaneously Zeno." That is, the system displays the following behavior: when the guard $\overline{G}_{e_s}$ is reached, there is a perfectly plastic impact $\overline{R}_{e_s}$ which causes the execution to land directly on the guard $\overline{G}_{e_z}$ and thus there is an instantaneous transition to the post-Zeno domain $\overline{D}_z$. This will imply, for example, that periodic orbits for systems of this form are 3-periodic.

Periodic orbits of completed systems. Let $\overline{\mathscr{SH}}^{\varepsilon}_{\mathbf{L}}$ be a completed SLHS.

In the special case of plastic impacts ($\varepsilon = 0$), a *plastic periodic orbit* is an execution $\overline{\chi}$ of $\overline{\mathscr{SH}}^{0}_{\mathbf{L}}$ with initial condition $x^* = c_0(t_0) \in \overline{D}_z$ satisfying:

- $\Lambda = \mathbb{N}$,
- $\lim_{i\to\infty} t_i - t_0 = \infty$,
- $\rho(i) = \begin{cases} z & \text{if} \quad i = 0, 3, 6, 9, \ldots \\ p & \text{if} \quad i = 1, 2, 4, 5, \ldots \end{cases}$,
- $c_{3i}(t_{3i}) = c_{3(i+1)}(t_{3(i+1)})$.

The *period* of the orbit is $T = t_3 - t_0$.

For non-plastic impacts ($\varepsilon > 0$), a *Zeno periodic orbit* is an execution $\overline{\chi}$ of $\overline{\mathscr{SH}}^{\varepsilon}_{\mathbf{L}}$ with initial condition $x^* = c_0(t_0) \in \overline{D}_z$ satisfying:

- $\Lambda = \mathbb{N}$,
- $\lim_{i\to\infty} t_i - t_0 = t_\infty < \infty$,
- $\rho(0) = z$ and $\rho(i) \equiv p$ for all $i \geq 1$,
- $x_\infty = \lim_{i\to\infty} c_i(t_i) = c_0(t_0) = x^*$.

The *period* of the orbit is $T = t_\infty$.

Stability of hybrid periodic orbits. We now define the stability of plastic periodic orbits. Note that we also could define the stability of Zeno periodic orbits, but as this definition is sufficiently more complicated and not necessary to the results presented here, we restrict our attention to the definition of the stability of plastic periodic orbits.

Definition 8. *A plastic periodic orbit $\overline{\chi}$ of $\overline{\mathscr{SH}}^{0}_{\mathbf{L}}$ with initial condition $x^* \in \overline{D}_z \subset \mathcal{Z}$ is* locally exponentially stable *if there exists a neighborhood $X \subset \overline{D}_z$ of x^* and positive constants M and $\mu \in (0,1)$ such that for any initial condition $x_0 = c_0(t_0) \in X$, the execution $\overline{\chi}'$ with this initial condition satisfies $\|c'_{3k}(t_{3k}) - x^*\| \leq M\|x_0 - x^*\|\mu^k$ for $k \in \mathbb{N}$.*

4.1 Main Result

The main result of this paper is that if there exists an exponentially stable plastic periodic orbit, then there exist Zeno periodic orbits for small coefficients of restitution. It is important to note that this result is in the spirit of [22] with three major differences: (1) it is more general in that we do *not* require $M \leq 1$ as was the case in [22] allowing it to be applied to bipedal robotic controllers which usually do not satisfy the assumption that $M \leq 1$, (2) the conditions of the theorem are easier to verify from a computational perspective (again, important for bipedal robots), and (3) the techniques used to prove the result are fundamentally different.

Theorem 2. *Let $\overline{\mathscr{SH}}^{0}_{\mathbf{L}}$ have a plastic periodic orbit $\overline{\chi}$ with initial condition $x^* = (q^*, \dot{q}^*) \in \overline{D}_z \subset \mathcal{Z}$ that is locally exponentially stable. If $\ddot{h}(q^*, \dot{q}^*) < 0$, then there exists a positive constant r such that for any coefficient of restitution $0 < \varepsilon < r$ there exists a Zeno periodic orbit of $\overline{\mathscr{SH}}^{\varepsilon}_{\mathbf{L}}$.*

The proof of this theorem will rely extensively on Poincaré maps associated to periodic orbits in hybrid systems (space constraints prevent a detailed introduction, but a complete definition can be found in [28]). For a completed hybrid system, let $\varphi^i_\tau(x) = \varphi^i(\tau^i(x), x)$, for $i = p, z$, be the flow associated to the vector field $\overline{f}_i$, where here $\tau^i : \overline{R}_{e_i}(\overline{G}_{e_i}) \to \mathbb{R}$ is the *time to impact function* which gives the time it takes to reach a guard from the image of another guard.

For a plastic periodic orbit, the Poincaré map is the partial function given by: $P : \mathcal{Z} = \overline{G}_{e_z} \to \mathcal{Z}$, where $P(x) = \overline{R}_{e_s}(\varphi^p_\tau(\overline{R}_{e_p}(\varphi^z_\tau(x))))$. The fixed point of the Poincareé map, $x^* = P(x^*)$ is just the point at which the periodic orbit intersects the surface $\mathcal{Z}$, i.e., it is simply the point x^* given in Def. 8. As with smooth dynamical systems [25], the (exponential) stability of a plastic periodic orbit (or periodic orbits in hybrid systems in general [20]) is equivalent to the (exponential) stability of the discrete-time dynamical system obtained through the Poincaré map, $x_{k+1} = P(x_k)$, at the equilibrium point x^*. This can be best understood by noting that in Def. 8 $c'_{3k}(t_{3k}) = P(c'_{3(k-1)}(t_{3(k-1)}))$.

PROOF. Since $x^* \in \overline{D}_z$, it is by definition a Zeno equilibrium point, and because $\ddot{h}(q^*, \dot{q}^*) < 0$ it is bounded-time locally Zeno stable by Thm. 1. Let W and U be the neighborhoods in G_h given in Def. 6. Note that here we implicity assume (without loss of much generality) that both W and U do not intersect $\overline{G}_{e_p}$ (this condition can be guaranteed as long as x^* is not in $\overline{G}_{e_p}$ and the coefficient of restitution is picked to be sufficiently small); that is, we assume that the Zeno point is reached before foot-strike, or that knee-lock occurs before foot-strike. In addition, let X be the neighborhood of x^* in $\mathcal{Z}$ given in Def. 8 which exists due to the assumption of local exponential stability of the plastic periodic orbit. Note that we can suppose[6] that for any execution $\overline{\chi}^0$ or $\overline{\chi}^\varepsilon$ of $\overline{\mathscr{SH}}^0_{\mathbf{L}}$ or $\overline{\mathscr{SH}}^\varepsilon_{\mathbf{L}}$, respectively, with initial condition $x_0 \in X$, it follows that $c_1^0(t_2) = c_1^\varepsilon(t_2) \in W$. Finally, for the sake of notational simplicity, assume that $x^* = 0$.

Since $\overline{\chi}$ is a locally exponentially stable plastic periodic orbit, P is locally exponentially stable at 0, and because P is smooth (since it is given by composing smooth functions), it follows by the converse Lyapunov theorem for discrete-time dynamical systems (see [11]) that there exists a function $V : X \to \mathbb{R}$ satisfying

$$c_1||x||^2 \leq V(x) \leq c_2||x||^2 \quad (14)$$

$$\Delta V(x) = V(P(x)) - V(x) \leq -c_3||x||^2 \quad (15)$$

$$|V(x) - V(y)| \leq c_4||x - y||(||x|| + ||y||) \quad (16)$$

for all $x, y \in X$ and positive constants c_1, c_2, c_3 and c_4. Let $\alpha = \min_{x\in\partial X} V(x)$, and take $\beta \in (0, \alpha)$. Then the set $\Omega_\beta = \{x \in X : V(x) \leq \beta\}$ is in the interior of X and is invariant under P since $\Delta V(x) < 0$, i.e., if $x_0 \in \Omega_\beta$ then $P(x_0) \in \Omega_\beta$.

The goal is to show that if an execution of $\overline{\mathscr{SH}}^\varepsilon_{\mathbf{L}}$ has an initial condition $x_0 \in \Omega_\beta$ then the Zeno point $x^\varepsilon_\infty(x_0) \in \Omega_\beta$ for a sufficiently small coefficient of restitution. To see this, let $y_0 = c_1^0(t_2) = c_1^\varepsilon(t_2)$ wherein it follows that $x^\varepsilon_\infty(x_0) = x^\varepsilon_\infty(y_0)$ and $P(x_0) = x^0_\infty(y_0)$. From (15) and (16),

$$\begin{aligned}
&V(x^\varepsilon_\infty(x_0)) - V(x_0)\\
&= V(x^\varepsilon_\infty(x_0)) - V(P(x_0)) + V(P(x_0)) - V(x_0)\\
&\leq |V(x^\varepsilon_\infty(x_0)) - V(P(x_0))| + V(P(x_0)) - V(x_0)\\
&\leq c_4||x^\varepsilon_\infty(y_0) - x^0_\infty(y_0)||(||x^\varepsilon_\infty(y_0)|| + ||x^\varepsilon_\infty(y_0)||)\\
&\qquad -c_3||x_0||^2.
\end{aligned}$$

Now, by Prop. 1, it follows that $||x^\varepsilon_\infty(y_0) - x^0_\infty(y_0)||$ goes to zero continuously as $\varepsilon \to 0$. As a result, for all $0 < \varepsilon < r$ (with r sufficiently small), $V(x^\varepsilon_\infty(x_0)) - V(x_0) \leq 0$. Therefore, $x_0 \in \Omega_\beta$ implies that $V(x_0) \leq \beta$ and $V(x^\varepsilon_\infty(x_0)) \leq V(x_0) \leq \beta$ so $x^\varepsilon_\infty(x_0) \in \Omega_\beta$.

We have established that Ω_β is invariant under the continuous[7] map $x^\varepsilon_\infty : \Omega_\beta \to \Omega_\beta$ associating to a point its Zeno point. By the fixed point theorem [10], there exists a fixed point x_z such that $x^\varepsilon_\infty(x_z) = x_z$. By the definition of Zeno periodic orbits, this implies the existence of such an orbit, i.e., the execution of $\overline{\mathscr{SH}}^\varepsilon_{\mathbf{L}}$ with initial condition x_z. □

[6]This supposition can be enforced by considering a subset of X if needed.

[7]Continuity is guaranteed as long as the neighborhood X does not intersect the guard $\overline{G}_{e_p}$; space constraints prevent a proof of this fact, but the reasoning is similar to the justification of continuity for hybrid systems with a single constraint [5].

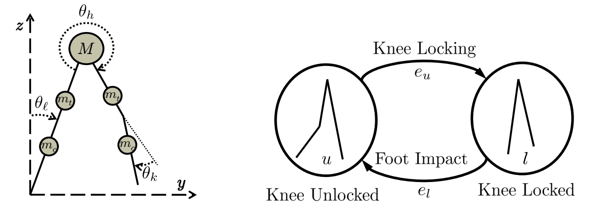


Figure 2: The configuration space of the 2D biped with knees (left) and a graphical representation of the domains of the hybrid system $\mathscr{H}^{\mathrm{B}}$ (right).

5. BIPEDAL MODEL WITH KNEE-LOCK

We begin by studying the case when a bipedal robot has knees that lock, i.e., knees in which the impact is perfectly plastic; this could be achieved physically by, for example, using knees with mechanical catches [17]. This section introduces the hybrid model for this system along with controllers that result in stable walking. The specific control laws that will be used are based upon the idea of *controlled symmetries* which mimic the gait of passive walkers walking down shallow slopes [16, 12, 6] by shaping the potential energy of the system [26]. The biped that will be considered has been studied in both 2D and 3D in [2].

The model of interest is a controlled bipedal robot with knees that walks on flat ground for which we will explicitly construct the hybrid system:

$$\mathscr{H}^{\mathrm{B}} = (\Gamma^{\mathrm{B}}, D^{\mathrm{B}}, G^{\mathrm{B}}, R^{\mathrm{B}}, F^{\mathrm{B}}).$$

We now introduce the elements of this hybrid system in a step-by-step fashion.

Discrete Structure The discrete structure for the model is given by $\Gamma^{\mathrm{B}} = (\{u, l\}, \{e_u = (u, l), e_l = (l, u)\})$. That is, there are two domains u, l and two edges e_u, e_l (see Fig. 2). In the first domain the biped's non-stance knee is unlocked and in the second domain the biped's knee is locked. Transitions occur from domain u to domain l when the knee locks, and from l to u when the foot strikes the ground. Note that the discrete structure of this model enforces temporal ordering to events (kneelock and footstrike) and this discrete structure implies that $D^{\mathrm{B}} = \{D^{\mathrm{B}}_u, D^{\mathrm{B}}_l\}$, $G^{\mathrm{B}} = \{G^{\mathrm{B}}_{e_u}, G^{\mathrm{B}}_{e_l}\}$, $R^{\mathrm{B}} = \{R^{\mathrm{B}}_{e_u}, R^{\mathrm{B}}_{e_l}\}$ and $F^{\mathrm{B}} = \{f^{\mathrm{B}}_u, f^{\mathrm{B}}_l\}$.

Configuration space and Lagrangian. Consider the configuration $Q^{\mathrm{B}} = \mathbb{R}^3$ with coordinates $q = (\theta_l, \theta_h, \theta_k)$, where θ_l is the angle of the leg from vertical, θ_h is the angle of the hip and θ_k is the angle of the knee (see Fig. 2). The Lagrangian for this system is of the form:

$$L^{\mathrm{B}}(q, \dot{q}) = \frac{1}{2}\dot{q}^T M^{\mathrm{B}}(q)\dot{q} - V^{\mathrm{B}}(q)$$

which is computed in the standard way.

Unilateral constraints. We will consider two unilateral constraints for this system. The first unilateral constraint enforces the knee being unlocked. It is therefore given by: $h^{\mathrm{B}}_u(q) = \theta_k$. The second unilateral constraint is the constraint that the foot is above the ground, and it is therefore given by:

$$h^{\mathrm{B}}_l(q) = 2\ell \cos\left(-\theta_l - \frac{1}{2}\theta_h\right) \cos\left(\frac{1}{2}\theta_h\right),$$

with ℓ the length of the leg.

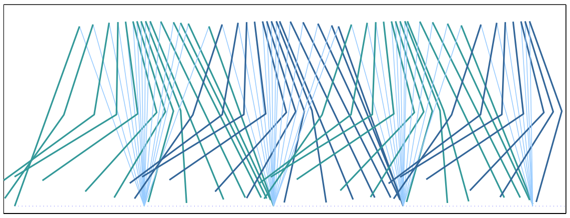


Figure 3: A walking gait of the 2D biped with plastic impacts at the knee.

Domain 1 (knee unlocked). The domain $D_u^{\rm B}$ is obtained as in (7) from $h_u^{\rm B}$. The vector field $f_u^{\rm B}$ is obtained as in (4) from the Lagrangian $L^{\rm B}$ and the feedback control law:

$$\Upsilon^{\rm B}(q,\dot{q}) = \frac{\partial V^{\rm B}}{\partial q}(q) - \frac{\partial V^{\rm B}}{\partial q}(q + (\gamma,0,0)^T)$$

where γ is a control gain that can be viewed as the "slope" that would yield walking for passive biped walking down a slope of γ radians. Note that applying this control law implies that $f_u^{\rm B}$ is just the dynamical system associated with the Lagrangian:

$$L_\gamma^{\rm B}(q,\dot{q}) = \frac{1}{2}\dot{q}^T M^{\rm B}(q)\dot{q} - V^{\rm B}(q + (\gamma,0,0)^T)$$

with no feedback control law. Also note that this control law uses full actuation at all joints, including the knee.

Domain 2 (knee locked). The domain $D_l^{\rm B}$ is obtained as in (7) from $h_l^{\rm B}$. The vector field $f_l^{\rm B}$ is obtained by imposing the holonomic constraint $\eta = h_l^{\rm B}$ in $f_u^{\rm B}$ (as outlined in (6)); this enforces the condition that the knee is locked in this domain. In particular, the control law in this domain is again the controlled symmetries control law enforced in Domain 1 since we use the vector field $f_u^{\rm B}$ (which includes this control law) to obtain the constrained dynamics on this domain. This implies that there is again a torque at the knee.

Edge 1 (knee locking). The guard $G_{e_u}^{\rm B}$ is obtained as in (7) from $h_u^{\rm B}$. The reset map $R_{e_u}^{\rm B}$ is also obtained from $h_u^{\rm B}$ as in (8). Note that for this model we assume the knee impact is *perfectly plastic*, i.e., $\varepsilon = 0$. The rest of this paper will be devoted to understanding what happens if this assumption is not satisfied.

Edge 2 (Foot impact). The guard $G_{e_l}^{\rm B}$ is obtained as in (7) from $h_l^{\rm B}$. The reset map $R_{e_l}^{\rm B}$ models a perfectly plastic impact at the foot which also relabels the stance and non-stance leg to account for the two legs switching roles. This is obtained through the same process outlined in [9], but space constraints prevent the inclusion of this equation.

Walking gait. For the model under consideration: $m_c = 0.05$kg, $m_t = 0.5$kg, $M_h = 0.5$kg, $\ell = 1$m, $r_c = 0.372$m, $r_t = 0.175$m. For a control gain of $\gamma = 0.0504$, the result is stable walking, i.e., there is a stable periodic orbit (see Fig. 5). For the purpose of illustration, the walking gait for these parameters and control gain can be seen in Fig. 3.

6. BIPEDAL MODEL WITH KNEE-BOUNCE

We now consider the case when the assumption of perfectly plastic impacts at the knee does not hold, i.e., the

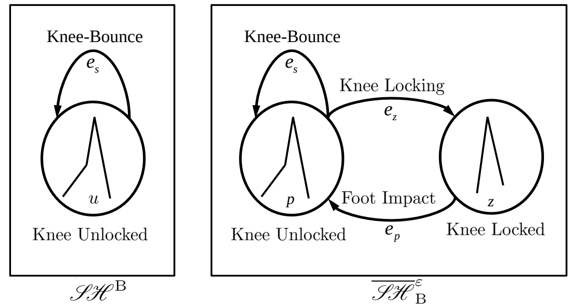


Figure 4: A graphical representation of (a) the simple hybrid system modeling knee-strike with a non-plastic impact, i.e., knee-bounce, and (b) the completed hybrid system in which the knee locks after the Zeno point is reached.

case where there is knee-bounce. Relaxing this assumption implies that the transition to domain 2 (knee locked) never formally takes place since this would involve an infinite number of discrete jumps, i.e., there is Zeno behavior. Therefore, relaxing this assumption results in a completely different hybrid system.

Non-plastic impacts at the Knee. We now relax the assumption that $\varepsilon = 0$ for the reset map obtained from the impact equations for knee impact.

Consider the hybrid Lagrangian $\mathbf{L}_u^{\rm B} = (Q^{\rm B}, L^{\rm B}, h_u^{\rm B})$ with $Q^{\rm B}$, $L^{\rm B}$ and $h_u^{\rm B}$ defined as in Sect. 5. From this hybrid Lagrangian we obtain a simple Lagrangian hybrid system

$$\mathscr{SH}^{\rm B} = (D_u^{\rm B}, G_{e_u}^{\rm B}, R_{h_u^{\rm B}}, f_u^{\rm B}), \tag{17}$$

where $D_u^{\rm B}$, $G_{e_u}^{\rm B}$ and $f_u^{\rm B}$ are the same continuous domain, guard and vector field for Domain 1 and Edge 1 of the hybrid system $\mathscr{H}^{\rm B}$ given in Sect. 5. The reset map $R_{h_u^{\rm B}}$ is obtained from $h_u^{\rm B}$ as given in (8) where now $0 < \varepsilon < 1$, i.e., it is a non-plastic impact (that also is not allowed to be perfectly elastic). See Fig. 4(a) for a graphical representation of this model, where the discrete transition occurs at *knee strike* not *knee lock*, i.e., the knee never locks because the impacts are non-plastic.

Zeno Behavior. Due to the non-plastic impacts, the hybrid system $\mathscr{SH}^{\rm B}$ is Zeno (by Thm. 1).

First, due the simple form of the unilateral constraint function on this domain ($h_u^{\rm B}(q) = \theta_k$), the set of Zeno equilibra is $\mathcal{Z}^{\rm B} = \{(q,\dot{q}) \in \mathbb{R}^6 : \theta_k = 0 \text{ and } \dot{\theta}_k = 0\}$. That is, the Zeno equilibria are the set of points such that the knee angle is zero with zero velocity, i.e., the set of points where the leg is straight.

To check for Zeno behavior, it is necessary to consider $\ddot{h}_u^{\rm B}$, which in this case is given by: $\ddot{h}_u^{\rm B}(q,\dot{q}) = (f_u^{\rm B}(q,\dot{q}))_6$. Note that this is a complex function, and so it is not possible to give a simple characterization of the points $(q^*,\dot{q}^*) \in \mathcal{Z}^{\rm B}$ such that $\ddot{h}_l^{\rm B}(q^*,\dot{q}^*) < 0$.

Completion of bipedal model. From the hybrid system $\mathscr{SH}^{\rm B}$ we obtain a completed hybrid system $\overline{\mathscr{SH}}^{\varepsilon}_{\rm B}$, where ε is the coeficent of resolution. This is given by "combining" the hybrid system $\mathscr{H}^{\rm B}$ given in Sect. 5 and the simple hybrid system $\mathscr{SH}^{\rm B}$ (Fig. 4(b)). Let $\overline{\mathscr{SH}}^{\varepsilon}_{\rm B} = (\overline{\Gamma}, \overline{D}, \overline{G}, \overline{R}, \overline{F})$, where $\overline{\Gamma}$ is given as in Def. 7 and

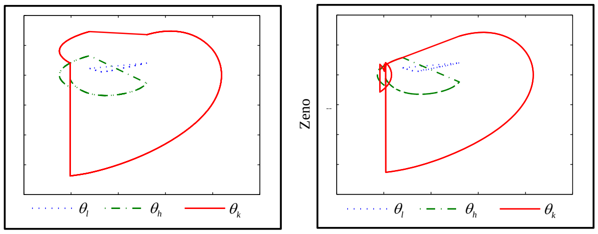


Figure 5: The periodic orbit associated to the walking gait for $\mathscr{H}^{\mathrm{B}}$, i.e., walking with knee-lock, which is equivalent to the plastic periodic orbit for $\overline{\mathscr{SH}}^0_{\mathrm{B}}$ (left). The Zeno periodic orbit for $\overline{\mathscr{SH}}^{\varepsilon}_{\mathrm{B}}$ with $\varepsilon = 0.25$, i.e., walking with knee-bounce (right).

- $\overline{D} = \{\overline{D}_p, \overline{D}_z\}$ where $\overline{D}_p = D^{\mathrm{B}}_u$ and $\overline{D}_z = D^{\mathrm{B}}_l$,
- $\overline{G} = \{\overline{G}_{e_s}, \overline{G}_{e_z}, \overline{G}_{e_p}\}$ where $\overline{G}_{e_s} = G^{\mathrm{B}}_{e_u} \backslash \mathcal{Z}^{\mathrm{B}}$, $\overline{G}_{e_z} = \mathcal{Z}^{\mathrm{B}}$ and $\overline{G}_{e_p} = G^{\mathrm{B}}_{e_l}$,
- $\overline{R} = \{\overline{R}_{e_s}, \overline{R}_{e_z}, \overline{R}_{e_p}\}$ where $\overline{R}_{e_s} = R_{h^{\mathrm{B}}_u}$ (which depends on ε), $\overline{R}_{e_z} = I$ and $\overline{R}_{e_p} = R^{\mathrm{B}}_{e_l}$,
- $\overline{F} = \{\overline{f}_p, \overline{f}_z\}$ where $\overline{f}_p = f^{\mathrm{B}}_u$ and $\overline{f}_z = f^{\mathrm{B}}_l$.

Note that $\overline{\mathscr{SH}}^0_{\mathrm{B}}$ and $\mathscr{H}^{\mathrm{B}}$ have the same qualitative behavior although they have slightly different structures. That is, the completion of $\mathscr{SH}^{\mathrm{B}}$ when $\varepsilon = 0$ is just the hybrid system $\mathscr{H}^{\mathrm{B}}$. We are, of course, interested in what happens when the assumption that $\varepsilon = 0$ is relaxed for the biped and its effect on walking gaits.

Application of Theorem 2 and Simulation Results. We now apply Thm. 2 to the completed hybrid system modeling the biped with non-plastic impacts at the knee $\overline{\mathscr{SH}}^{\varepsilon}_{\mathrm{B}}$ to show that a plastic periodic orbit for $\varepsilon = 0$ implies the existence of a Zeno periodic orbit for $\varepsilon > 0$, i.e., that walking with knee-lock implies walking with knee-bounce when the knee-bounce is sufficiently small.

Due to the equivalence of $\overline{\mathscr{SH}}^0_{\mathrm{B}}$ and $\mathscr{H}^{\mathrm{B}}$, and since there was a periodic orbit for $\mathscr{H}^{\mathrm{B}}$, there is a plastic periodic orbit for $\overline{\mathscr{SH}}^0_{\mathrm{B}}$ as pictured in Fig. 5. The exponential stability of this control law can be verified by considering the Poincaré map; the exponential stability of this map implies the exponential stability of the plastic periodic orbit. Moreover, the exponential stability of the Poincaré map can be verified by considering the eigenvalues of its linearization and ensuring that none have magnitude greater than 1. In this case, the largest eigenvalue has magnitude 0.7329 indicating exponential stability of the plastic periodic orbit. Finally, the value of $\ddot{h}$ at the Zeno equilibria point is $\ddot{h}(x^*) = -50.135$. Therefore, the assumptions of Thm. 2 are satisfied.

As a result of Thm. 2, there exists a Zeno periodic orbit for $\overline{\mathscr{SH}}^{\varepsilon}_{\mathrm{B}}$ for a range of coefficients of restitution $0 < \varepsilon \leq r$. Of course, there is not an explicit value for r stated in the theorem, but we were able to find a rather large range of coefficients of restitution resulting in Zeno periodic orbits. One of these orbits can be seen in Fig. 5 for $\varepsilon = 0.25$. The effect of Zeno behavior on the biped can be clearly seen in this figure due to the "bouncing" behavior of θ_k, i.e., the Zeno periodic orbit clearly displays knee-bounce. The effect of knee-bounce on the behavior of the biped can be better seen when comparing the positions and velocities of the knee

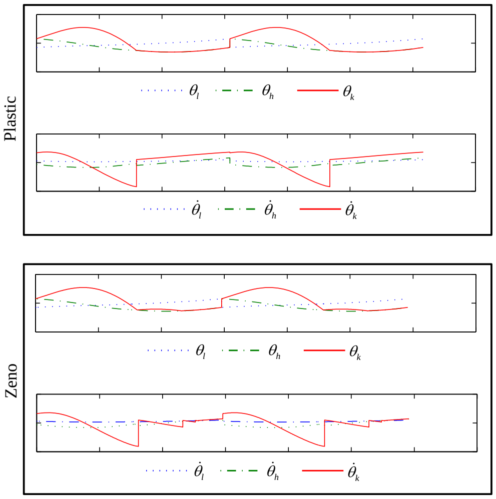


Figure 6: The positions and velocities over time for the walking gait in the case of plastic impacts (left) and non-plastic impacts (right) at the knee.

over time in the case of a zero and nonzero coefficient of restitution as seen in Fig. 6.

It is important to note that, although Thm. 2 implies the existence of a Zeno periodic orbit, it does not give any guarantees on the stability of this orbit. While a formal result of this nature would be interesting, for all practical purposes the stability of the Zeno periodic orbit can be checked the same way that the stability of the plastic periodic orbit was checked: by numerically computing the eigenvalues of the linearization of the Poincaré map. In the case of the Zeno periodic orbit in Fig. 5 we find that the largest eigenvalue has magnitude 0.2245, implying that we in fact get a stable walking gait in the case of knee-bounce.

Videos of the walking gait with both knee-lock and knee-bounce can be found at [1]. It is interesting to compare these walking gaits with the behavior of the passive dynamic walking with knees by McGeer (videos of this can also be found at [1]). In the case of knee lock, it can be seen that the behavior of the simulated and actual robotic walking are very similar. In the case of knee-bounce, we postulate that the McGeer biped falls due to there being a larger coefficient of restitution associated with knee-bounce than is the case with the simulated biped with knee-bounce, i.e., we found similar behavior in the simulated system when ε was taken to be larger than about 0.4. Yet, despite the differences in the coefficient of restitution between the physical and simulated system, the similarity between knee-bounce in simulation and reality is quite remarkable. This indicates that models with Zeno behavior can effectively simulate real physical systems in order to say useful things about the behavior of these systems.

7. CONCLUSION

Motivated by the issue of knee-bounce in bipedal robotic walking, this paper shows that knee-bounce may not always negatively affect the stability of bipedal walking as long as the bounce is kept sufficiently small. This is demonstrated through the observation that knee-bounce in walking is just an example of Zeno behavior in hybrid systems. Conditions on when Zeno behavior exists are used to characterize the difference between orbits in hybrid systems with plastic and non-plastic impacts. With this in hand, the notion of generalized completion of a hybrid system is introduced, extending the traditional notion of completion to a setting that allows bipedal robots to be modeled with this formalism. The main result of this paper is that if a plastic periodic orbit is stable (the biped has stable walking with knee-lock) then under easily verifiable conditions a Zeno periodic orbit exists (the biped has stable walking with knee-bounce as long as the knee-bounce is sufficiently small). These results are applied to a specific example of a bipedal robot with knees, and walking gaits are presented in the case of both knee-lock and knee-bounce.

Since this paper considered Zeno behavior that occurs at the knee, the natural question to ask is: what happens if Zeno behavior occurs at the foot, or the foot and knee simultaneously? Addressing this question will be surprisingly difficult due to the differences between the impact equations at the knee and foot. At the knee, the impacts are a result of unilateral constraints, and impacts related to these types of constraints have been well-studied; it was by building upon previous results from the author and other researchers that the main results of these paper were able to be shown. This preexisting work was non-trivial, taking years to establish. The first step in extending this work to more interesting types of impacts, such as those that occur at the feet, is to study Zeno behavior in the context of these impacts.

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