# Rank Deficiency and Superstability of Hybrid Systems with Application to Bipedal Robots

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***Abstract*— The objectives of this paper are to study the rank properties of flows of hybrid systems and show that they are fundamentally different from those of smooth dynamical systems, and consider applications that emphasize the importance of these differences. It is well known that the flow of a smooth dynamical system has rank equal to the space on which it evolves. We prove that, in contrast, the rank of a solution to a hybrid system, a *hybrid execution*, is always less than the dimension of the space on which it evolves and falls within easily-computed and possibly distinct upper and lower bounds. Our main contribution is the derivation of conditions for when an execution fails to have maximal rank, *i.e.*, when it is rank deficient. Given the importance of periodic behavior in many hybrid systems applications, for example in bipedal robots, these conditions are applied to the special case of periodic hybrid executions. Our secondary contribution is the derivation of *superstability* conditions for when a periodic execution is rank deficient and has rank equal to $0$, that is, we determine when the execution is completely insensitive to perturbations in initial conditions. The results of this paper are illustrated on two applications, one of which is the classical single-domain, planar compass biped.**

## I. Introduction

Hybrid systems consist of both continuous and discrete components and, as such, are capable of modeling a wide variety of physical systems, *i.e.*, systems that evolve with both continuous and discrete dynamics. Although hybrid systems model a wide variety of applications, we may not in general assume that they share the same fundamental properties as smooth dynamical systems. Moreover, the interaction of the smooth and discrete components of a hybrid system can result in solution behavior that is impossible for smooth dynamical systems to exhibit. For example, the existence and uniqueness properties of solutions of hybrid systems — called *hybrid executions* — are not the same as for smooth systems [1], [2]; therefore, one may not regard the stability of hybrid system equilibria in the same way as the stability of smooth system equilibria [3]. An example of the fundamental difference between the solutions of smooth and hybrid systems is *Zeno behavior*, where under certain conditions an execution of a hybrid system can take an infinite number of discrete transitions in a finite amount of time [4]. Recent work [5] has also shown that Poincaré maps for hybrid systems are fundamentally different from Poincaré maps for smooth systems.

This research is supported in part by NSF award CNS-0953823 and NHARP Award 000512-0184-2009.

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The first contribution of the present work is the extension of the results in [5] to arbitrary, non-periodic hybrid executions. In particular, we show that the rank of an execution will always fall between possibly distinct upper and lower bounds, and that the upper bound is always less than the dimension of the space on which the execution evolves. This result is in marked contrast with smooth dynamical systems, where the rank of a solution is strictly equal to the dimension of the space. Our main contribution, however, is the derivation of conditions describing when an execution fails to have maximal rank, that is, when it is *rank deficient*. The secondary contribution of this work emerges from application of the main result to periodic solutions of hybrid systems. We show that when an execution is periodic *and* rank deficient it may be possible for the system to be *superstable*. Recall that a discrete dynamical system is said to be superstable when it is completely insensitive to perturbations in initial conditions [6]. This occurs when the linearization of the discrete dynamical system is equal to $0$ at a superstable equilibrium point. By considering superstability from within the context of rank deficiency of executions, we obtain a condition describing when a periodic hybrid execution is completely insensitive to perturbations in its initial conditions.

We foresee that the superstability conditions presented here could enable the design of controllers that reduce the sensitivity of hybrid systems to perturbations. In [7], finite-time controllers and the properties of feedback-linearized systems are used to reduce the stability analysis of a planar biped to an interval of the real line. We foresee that, in analogy to this work, our rank deficiency conditions could be used to enable the design of (feedback-linearizing) controllers that reduce the stability analysis of complex hybrid systems to lower-dimensional spaces.

Finally, it is worth noting that, to the authors' knowledge, there is no prior literature that directly addresses the fundamental rank properties of hybrid executions, nor the implications of rank deficiency to the periodic stability of hybrid systems.

We begin our analysis with a review of the standard theory of smooth dynamical systems in Section II. We show that the continuous-time flow of a smooth vector field can be converted into a discrete map [8], [9]. This standard theory applies directly to the smooth components of a hybrid system, leading to straightforward techniques for linearizing executions of hybrid systems, in Section III. Using the linearization of a hybrid system we determine the rank of arbitrary hybrid executions and derive necessary and sufficient conditions for the rank of an execution to fall below its upper bound. These are the rank deficiency conditions. In Section IV we specialize the rank deficiency conditions to periodic hybrid systems and their executions, called hybrid periodic orbits, and consider two separate applications. In the first, we illustrate a necessary and sufficient condition for a hybrid periodic orbit to be superstable. In the second, we consider the 2-link planar compass biped and use its linearization to accurately detect the occurrence of a period-doubling bifurcation.

## II. Smooth dynamical systems

In this section we review standard results [8] on the trajectories of smooth dynamical systems that will be necessary to our analysis of hybrid systems in Section III. In particular we review how to convert the flow, which depends continuously on time, into a discrete map. When the flow is a closed orbit this conversion results in the well-known results on periodic stability of smooth dynamical systems [9].

A smooth dynamical system is a tuple $(M, f)$, where $M$ is a smooth manifold with tangent bundle $TM$ and $f : M \to TM$ is a smooth vector field such that for the canonical projection map $\pi : TM \to M$, $\pi \circ f = \mathrm{Id}$, where Id is identity on $M$. We will assume that $M \subset \mathbb{R}^n$, in which case we can write the vector field in coordinates as $\dot{x} = f(x)$ with $x \in M \subset \mathbb{R}^n$ where necessarily $\dot{x} \in T_xM$. A smooth function $g : M \to N$ between manifolds induces a map between the tangent space $Dg(x) : T_xM \to T_{g(x)}N$; this is just the Jacobian or derivative.

### A. Flows and variational equations

The unique solution to the differential equation $\dot{x} = f(x)$ with initial condition $x_0 \in M$ is a trajectory $c : I \subset [0, \infty) \to M$ such that $c(t_0) = x_0$ if $I = [t_0, t_1]$, for some $t_1 > t_0$. We refer to this curve as an *integral curve* or *orbit* of $f(x)$. The *flow* of the smooth vector field $\dot{x} = f(x)$ is a smooth map $\phi : I \times U \to U' \subset M$, where $U$ is some neighborhood of $x_0 = c(t_0)$, satisfying the following properties for all $r, s, t \in I$,

$$c(t_0) = \phi_0(c(t_0))$$
$$c(t_0 + t + s) = \phi_{t+s}(c(t_0)) = \phi_t \circ \phi_s(c(t_0))$$
$$\phi_{-r} \circ \phi_r(x_0) = x_0 \Rightarrow \phi_{-r} = (\phi_r)^{-1}$$

The flow, with $t$ considered a parameter, is a diffeomorphism $\phi_t : U \to U'$, for all $t \in I$.

The *space derivative* or *fundamental matrix* of $\phi_t(x_0)$ is simply the partial derivative of the flow with respect to initial conditions, $D_x\phi_t(x_0) := \partial\phi_t(x_0)/\partial x_0$, (*cf.* [8]–[10]) and satisfies the time-varying, matrix-valued differential equation

$$\dot{\Phi}(t) = A(t)\,\Phi(t), \tag{1}$$

where $\Phi(t) = D_x\phi_t(x_0)$ and $A(t) := Df\big(\phi_t(x_0)\big)$. As an integral curve, $\Phi(t)$ is nonsingular for all $t$ and has the property that $\dot{\phi}_t(x_0) = \Phi(t)\Phi^{-1}(0)\,\dot{\phi}_0(x_0) = \Phi(t)\,\dot{\phi}_0(x_0)$. That is, with $x_1 = \phi_t(x_0)$, $f(x_1) = \Phi(t)\,f(x_0)$. Note in particular that $\Phi(0) = \mathrm{Id}_n$, the $n \times n$ identity matrix. In general, $\phi_t(x_0)$ and $\Phi(t)$ must be obtained by simultaneous numerical integration, as described in [10], [11].

### B. Flows to sections

We are interested in the case when the flow is allowed to evolve until it reaches a certain smooth hypersurface called a local section, which we may construct through any point of the flow that is not an equilibrium point [12].

*Definition 1:* A *local section* of a vector field $\dot{x} = f(x)$ on $M$ is a smooth codimension-1 submanifold $S$ of $M$ that is also *transverse* to the flow.

$$S = \{x \in M \mid h(x) = 0 \text{ and } L_f h(x) \neq 0\},$$

where $h : M \to \mathbb{R}$ is a $C^1$ function and $L_f h$ is the Lie derivative. More generally, any submanifold $N \subset M$ is said to be *transverse* to the flow (or vector field $f$) if $f(x)$ is not in $T_xN$.

The time it takes a flow to reach a local section from initial conditions is given by a well-defined map. We reproduce the following Lemma from [8], the proof of which follows from direct application of the implicit function theorem.

*Lemma 1 (Hirsch & Smale, 1974):* Let $S$ be a local section, $x_0 \in M$ and $x_1 = \phi_t(x_0) \in S$. There exists a unique, $C^1$ function $\tau : U_0 \to [0, \infty)$ called the *time-to-impact map* such that for $U_0$ a sufficiently small neighborhood of $x_0$, $\phi_{\tau(x)}(x) \in S$ for all $x \in U_0$.

The Lemma allows us to convert the flow to a section into a discrete map $\phi_\tau : U_0 \to V$ defined by $\phi_\tau(x) := \phi_{\tau(x)}(x)$ for all $x \in U_0$, where $U_0$ is defined as above and $V := \phi_\tau(U_0) \cap S$ is the image of $\phi_\tau$ in $S$. However, in general $\phi_\tau$ does not have the same rank as the flow $\phi_t$.

### C. Rank of flows to sections

The flow $\phi_t : U \to U'$, with $t$ considered a fixed parameter, is a diffeomorphism, so its total derivative, $D\phi_t$, will always have full rank. This is easily confirmed by computing $D\phi_t(x) = D_x\phi_t(x) = \Phi(t)$, which is nonsingular. The total derivative of the flow to a section $\phi_\tau : U_0 \to V \subset S$, on the other hand, is given [5], [8], [10] by

$$\begin{aligned} D\phi_\tau(x_0) &= \Phi(\tau(x_0)) + \dot{\phi}_\tau(x_0)D\tau(x_0) \\ &= \left(\mathrm{Id}_n - \frac{f(x_1)Dh(x_1)}{L_f h(x_1)}\right)\Phi(\tau(x_0)), \end{aligned} \tag{2}$$

where $x_0 \in U_0$, $x_1 = \phi_\tau(x_0)$, $\mathrm{Id}_n$ is the $n \times n$ identity matrix and $h$ defines the section $S$, as in Definition 1. It was also established in Theorem 1 and Corollary 1 of [5] that the flow to a section has rank equal to the dimension of the section and $\phi_\tau$ is therefore only a diffeomorpism between local sections. We summarize the generic properties of flows to sections:

- **(S1)** For any local section $S$ of $c(t_0)$ there exists a sufficiently small neighborhood $U_0$ of $c(t_0)$ such that $\phi_\tau(U_0) \subset S$.
- **(S2)** From Theorem 1 and Corollary 1 of [5] we know there exists a local section $S_0$ through $c(t_0)$ such that for $V_0 := U_0 \cap S_0$ and $V := \phi_\tau(U_0) \cap S$, the restricted map $\phi_\tau : V_0 \to V$ is a diffeomorphism with rank $n - 1$.

These properties will be revisited for hybrid systems.

## III. Hybrid dynamical systems

Our objective is to understand the rank properties of arbitrary hybrid executions in order to enable the design of controllers that improve the stability properties of hybrid systems. We begin by revisiting the results of the previous section from the perspective of hybrid systems and their executions.

### A. Hybrid systems and executions

*Definition 2:* A hybrid system is a tuple $\mathcal{H} = (\Gamma, D, G, R, F)$, where

- $\Gamma = (Q, E)$ is a graph such that $Q = \{q_1, \ldots, q_k\}$ is a set of $k$ vertices and $E = \{e_1 = (q_1, q_2), e_2 = (q_2, q_3), \ldots\} \subset Q \times Q$. With the set $E$ we define maps $\mathrm{sor} : E \to Q$ which returns the source of an edge, the first element in the edge tuple, and $\mathrm{tar} : E \to Q$, which returns the target of an edge or the second element in the edge tuple.
- $D = \{D_q\}_{q \in Q}$ is a collection of smooth manifolds called *domains*, where $D_q$ is assumed to be embedded submanifolds of $\mathbb{R}^{n_q}$ with $\dim(D_q) = n_q \geq 1$.
- $G = \{G_e\}_{e \in E}$ is a collection of *guards*, where $G_e$ is assumed to be an embedded submanifold of $D_{\mathrm{sor}(e)}$.
- $R = \{R_e\}$ is a collection of *reset maps* which are smooth maps $R_e : G_e \to D_{\mathrm{tar}(e)}$.
- $F = \{f_q\}_{q \in Q}$ is a collection of Lipschitz vector fields on $D_q$, such that $\dot{x} = f_q(x)$.

The continuous and discrete dynamics of a hybrid system are described using a notion of solution called a hybrid execution.

*Definition 3:* A *hybrid execution* is a tuple $\chi = (\Lambda, I, \rho, C)$, where

- $\Lambda = \{0, 1, 2, 3, \ldots\} \subseteq \mathbb{N}$ is an indexing set.
- $I = \{I_i\}_{i \in \Lambda}$ such that with $|\Lambda| = N$, $I_i = [t_i, t_{i+1}] \subset \mathbb{R}$ and $t_i \leq t_{i+1}$ for $0 \leq i < N-1$. If $N$ is finite then $I_{N-1} = [t_{N-1}, t_N]$ or $[t_{N-1}, t_N)$ or $[t_{N-1}, \infty)$, with $t_{N-1} \leq t_N$.
- $\rho : \Lambda \to Q$ is a map such that $e_{\rho(i)} := (\rho(i), \rho(i+1)) \in E$.
- $C = \{c_i\}_{i \in \Lambda}$ is a set of continuous trajectories where each $c_i$ is the integral curve of the vector field $f_{\rho(i)}$ on $D_{\rho(i)}$. Specifically, $c_i(t) = \phi^{\rho(i)}_{t-t_i}(c_i(t_i))$, where $\phi^{\rho(i)}_t$ is the flow associated with $f_{\rho(i)}$.

We require the consistency conditions:

- For $i < |\Lambda|$ and for all $t \in I_i$, $c_i(t_i) = \phi^i_0(c_i(t_i))$, $c_i(t) \in D_{\rho(i)}$ and $c_i(t_{i+1}) \in G_{e_{\rho(i)}}$.
- For $i < |\Lambda| - 1$, $R_{e_{\rho(i)}}(c_i(t_{i+1})) = c_{i+1}(t_{i+1})$.

*1) Assumptions:* We will only consider hybrid executions that are *deterministic* and *non-blocking* [1]. We further impose the following conditions on $\chi$ in order to ensure that the guards and reset maps are sufficiently "well-behaved." Let $i < |\Lambda| - 1$ and $e = (\rho(i), \rho(i+1))$.

- **(A1)** The execution does not have any equilibria, *i.e.*, $f_{\rho(i)}(c_i(t)) \neq 0$, for all $t \in I_i$.
- **(A2)** $R_e$ has constant rank $r_e$ and $R_e(G_e)$ is a submanifold of $D_{\mathrm{tar}(e)}$.
- **(A3)** $G_e$ is a section, *i.e.*, $\dim(G_e) = \dim(D_{\mathrm{sor}(e)}) - 1$ and $f_{\mathrm{sor}(e)}(x) \notin T_x G_e$ for all $x \in G_e$. Furthermore, $S^i \subset G_e$.
- **(A4)** $R_e(G_e)$ is transverse to $f_{\mathrm{tar}(e)}$ whenever $\dim(D_{\mathrm{sor}(e)}) \leq \dim(D_{\mathrm{tar}(e)})$, that is, $f_{\mathrm{tar}(e)}(y) \notin T_y R_e(G_e)$ for all $y \in R_e(G_e)$.

As we will see, assumption (**A4**) will allow us to tighten the lower bound on the rank of our executions.

*2) Properties:* We may extend properties (**S1-2**) of flows to sections from Section II to every integral curve $c_i \in C$, $i < |\Lambda|$, satisfying (**A1-4**).

- **(H1)** For any local section $S^i \subset G_e$ of $c_i(t_{i+1})$ there exists a sufficiently small neighborhood $U^i_0$ of $c_i(t_i)$ such that $\phi^{\rho(i)}_\tau(U^i_0) \subset S^i$.
- **(H2)** From Theorem 1 and Corollary 1 of [5] we know there exists a local section $S^i_0$ through $c_i(t_i)$ such that for $V^i_0 := U^i_0 \cap S^i_0$ and $V^i := \phi^{\rho(i)}_\tau(U^i_0) \cap S^i$, the restricted map $\phi^{\rho(i)}_\tau : V^i_0 \to V^i$ is a diffeomorphism with rank equal to $\dim(D_{\rho(i)}) - 1$.

These properties are generically satisfied by any flow that reaches a guard, and will be necessary to our results on rank deficiency in the following subsections.

*3) Fundamental hybrid executions:* The rank of a hybrid execution is determined by the rank of its linearization, or total derivative, at every point. This motivates the following definition.

*Definition 4:* The *fundamental hybrid execution* associated with a given execution $\chi$ is a tuple $\mathcal{F}\chi = (\Lambda, I, \rho, C, W)$ where $\Lambda$, $I$, $\rho$, and $C$ are given in Definition 3 and $W = \{\Phi_i\}_{i \in \Lambda}$ is a set of continuous matrix-valued trajectories. Every $\Phi_i$ is an integral curve of $\dot{\Phi}_i(t - t_i) = Df_{\rho(i)}(c_i(t))\, \Phi_i(t - t_i)$. Specifically, $\Phi_i(t - t_i) = D_x \phi^{\rho(i)}_{t-t_i}(c_i(t_i))$ is the fundamental matrix or space derivative of the flow evaluated along $c_i \in C$ and every $\Phi_i \in W$ has the property $\dot{\phi}^i_t(c_i(t_i)) = \Phi_i(t - t_i)\, \dot{\phi}^i_0(c_i(t_i))$.

As mentioned in Section II, in general, $f_{\rho(i)}$ and $\dot{\Phi}_i$ must be integrated simultaneously.

### B. Rank of hybrid executions

Let $\mathcal{H}$ be a hybrid system and $\chi$ its hybrid execution with initial condition in the guard, $c_0(t_0) \in G_{e_{\rho(0)}}$. We are interested in finding a function relating the initial condition to a point $c_i(t_{i+1})$ in the guard $G_{e_{\rho(i)}}$, for some $i < |\Lambda|$. In fact this relation is given by the *partial* function $\psi_{\rho(i)} : V^0 \to V^i$ defined by $c_i(t_{i+1}) = \psi_{\rho(i)}(c_0(t_0))$, with

$$\psi_{\rho(i)} = \phi^{\rho(i)}_\tau \circ R_{e_{\rho(i-1)}} \circ \cdots \circ \phi^{\rho(1)}_\tau \circ R_{e_{\rho(0)}}, \qquad (3)$$

and the neighborhoods $V^0$ and $V^i$ of $c_0(t_0)$ and $c_i(t_{i+1})$ defined as in (**H2**). We may think of the partial function as describing the progress of the execution through the hybrid system $\mathcal{H}$.

*Remark 1:* This sequence of discrete maps $\psi_{\rho(i)}$ is a partial function since there is no guarantee that all of the points in the image of each reset map reach the guard. More precisely, we could call $\psi_{\rho(i)}$ a function if we could guarantee that $R_{e_{\rho(j)}}(\phi^{\rho(j)}_\tau(U^j_0)) \subset U^{j+1}_0$ for every $j \leq$

$i-1 < |\Lambda|$. This condition can be met if each neighborhood $U_0^j$ is made sufficiently small.

Let $\mathcal{F}\chi$ be the fundamental execution associated with $\chi$. We compute the total derivative of $\phi_\tau^{\rho(j)}$ using (2), as follows. For ease of notation let $x_0 = c_j(t_j)$, $x_1 = c_j(t_{j+1})$ and $h_j : D_{\rho(j)} \to \mathbb{R}$ define the local section $S^j \subset G_{e_{\rho(j)}}$ as in Definition 1. Then,

$$\begin{aligned} D\phi_\tau^{\rho(j)}(x_0) &= \Phi_j(\tau(x_0)) + f_{\rho(j)}(x_0)\, D\tau(x_0) \qquad (4) \\ &= \left(\mathrm{Id}_{n_j} - \frac{f_{\rho(j)}(x_1)\, Dh_j(x_1)}{Dh_j(x_1)\, f_{\rho(j)}(x_1)}\right) \Phi_j(\tau(x_0)), \end{aligned}$$

where $\tau(x_0) = t_{j+1} - t_j$ is the time it takes the flow to reach the guard and $n_j = \dim(D_{\rho(j)})$ is the dimension of the manifold. For the sake of brevity, in the following discussion we will merely assume that $\mathcal{F}\chi$ is available when it is necessary to compute $D\phi_\tau^{\rho(j)}$.

*Remark 2:* At this point it is worth considering the case of a *trivial* hybrid system $\mathcal{H}$ that is equivalent to a smooth dynamical system. Necessarily, the reset maps of a trivial hybrid system are equal to identity, the vector fields are equal so that $f_q = f$ for all $q \in Q$, and the execution $\chi$ is equivalent to the continuous flow of a single vector field. For such hybrid systems the choice of local section is irrelevant because there is no mechanism forcibly restricting the flow to a transverse local section. This is, however, exactly what occurs for *non-trivial* hybrid systems, where a continuous flow is permitted to evolve on a domain only until it reaches a guard. Thus, although the results that follow certainly apply, they are not pertinent to the analysis of trivial hybrid systems. We therefore consider only non-trivial hybrid systems, which, as we will see, always have rank less than the dimension of the space on which they evolve.

Our analysis of the rank properties of $\chi$ is aided by identifying the terms in (3) that can be associated with each edge in the graph $\Gamma$ of $\mathcal{H}$.

*Definition 5:* Let $i < |\Lambda| - 1$. For every edge $e = (\rho(i), \rho(i+1)) \in E$, the *edge map* $\psi_e : V^i \to V^{i+1}$ takes the guard of one domain to the next and is defined $\psi_e = \phi_\tau^{\mathrm{tar}(e)} \circ R_e$. Using the edge map, (3) becomes

$$\psi_{\rho(i)} = \psi_{e_{\rho(i-1)}} \circ \cdots \circ \psi_{e_{\rho(0)}}, \qquad (5)$$

$$D\psi_{\rho(i)} = D\psi_{e_{\rho(i-1)}} \circ \cdots \circ D\psi_{e_{\rho(0)}}. \qquad (6)$$

Note in (6) that $D\psi_{\rho(i)}$ is the composition of $i$ linear maps. Let $\{A_i\}_{i=1}^k$ be a collection of $n_{i+1} \times n_i$ real-valued matrices. Repeated application of Sylvester's inequality shows that the composition $\prod_{i=1}^k A_i = A_1 \circ A_2 \circ \cdots \circ A_k$ is bounded above and below:

$$\mathrm{rank}\left(\prod_{i=1}^k A_i\right) \leq \min_{i \in \{1,\ldots,k\}} \{\mathrm{rank}(A_i)\}, \qquad (7)$$

$$\mathrm{rank}\left(\prod_{i=1}^k A_i\right) \geq \sum_{i=1}^k \mathrm{rank}(A_i) - \sum_{i=1}^{k-1} n_i. \qquad (8)$$

Recall the rank-nullity theorem [13]: for every linear map $A : \mathbb{R}^n \to \mathbb{R}^m$, $\mathrm{rank}(A) + \mathrm{nty}(A) = n$, where $\mathrm{nty}(A)$ is the dimension of the nullspace of $A$. The following Lemma is a consequence of rank-nullity. We omit the straightforward proof for the sake of brevity.

*Lemma 2:* Let $A$ and $B$ be linear maps. Then

$$\mathrm{nty}(B \circ A) - \mathrm{nty}(A) = \dim(\mathrm{ns}(B) \cap \mathrm{im}(A)).$$

The Lemma and the rank-nullity theorem allow us to compute the rank of the execution by determining the rank of every edge map in $\psi_{\rho(i)}$. For the sake of brevity we omit the proof of the following result.

*Lemma 3:* Let $i < |\Lambda| - 1$. For every edge $e = (\rho(i), \rho(i+1))$, the rank of the edge map $\psi_e : V^i \to V^{i+1}$ is bounded from below by $\mathrm{rank}(\psi_e) \geq \mathrm{rank}(R_e) - 1$.

However, if $R_e(G_e)$ is transverse to $f_{\mathrm{tar}(e)}$ then $\mathrm{rank}(\psi_e) = \mathrm{rank}(R_e)$.

*Remark 3:* Lemma 3 implies that the rank of an edge map is only known exactly when the transversality of $R_e(G_e)$ is guaranteed. Since we cannot determine the rank of all edge maps where transversality is not guaranteed, we cannot determine the exact rank of the execution. In order to obtain a tighter lower bound on the rank of the execution, we track the number of edges where $R_e(G_e)$ may not be transverse to $f_{\mathrm{tar}(e)}$. Because we only assume transversality when $\dim(D_{\mathrm{sor}(e)}) > \dim(D_{\mathrm{tar}(e)})$, we need only track the number of source domains that are greater in dimension than their target domains.

The following definitions allow us to track the progress of the execution through the graph $\Gamma$ of $\mathcal{H}$.

*Definition 6:* Given $i < |\Lambda| - 1$, the set of *traversed edges* is $E_i = \{e_{\rho(0)}, \ldots, e_{\rho(i-1)}\}$, and the set of *visited vertices* is the set of all source and target vertices of $E_i$,

$$Q_i = \mathrm{sor}(E_i) \cup \mathrm{tar}(E_i) = \{\rho(0), \ldots, \rho(i)\}.$$

*Definition 7:* Let $m$ be the number of *non-transverse edges* for which we do not assume $R_e(G_e)$ is transverse to $f_{\mathrm{tar}(e)}$. Then $m$ is given by

$$m = \left|\{e \in E_i : \dim(D_{\mathrm{sor}(e)}) > \dim(D_{\mathrm{tar}(e)})\}\right|.$$

We now show that the rank of an execution falls between possibly distinct upper and lower bounds. The following result is the extension of Theorem 4 in [5] to arbitrary, non-periodic hybrid systems and executions. We omit the proof for reasons of space[1].

*Theorem 4:* Let $\mathcal{H}$ be a hybrid system with execution $\chi$ satisfying assumptions (**A1-4**). For any $i < |\Lambda| - 1$,

$$\mathrm{rank}(\psi_{\rho(i)}) \leq \min_{e \in E_i} \{\mathrm{rank}(R_e)\} \leq \min_{q \in Q_i} \{\dim(D_q) - 1\},$$

$$\mathrm{rank}(\psi_{\rho(i)}) \geq \sum_{e \in E_i} \mathrm{rank}(R_e) - m - \sum_{q \in \mathrm{sor}(E_i) - \{\rho(0)\}} (\dim(D_q) - 1),$$

and where $m$, $E_i$ and $Q_i$ are given in Definitions 6 and 7.

If the upper and lower bounds on rank in Theorem 4 are distinct then there must be a mechanism that causes an execution to fail to have maximal rank. We determine this mechanism in the next section.

[1]Full proofs of Lemmas 2, 3 and Theorem 4 may be found in the preprint [14], available online for the purposes of review at `http://people.tamu.edu/~ericdbw/NAHS2011.pdf`.

### C. Rank deficiency of hybrid executions

Our objective is to understand the causes of rank deficiency. As we will see, rank deficiency can result in superstable hybrid systems that are completely insensitive to perturbations in initial conditions. We begin by formally defining the rank deficiency of a hybrid execution.

*Definition 8:* Let $\mathcal{H}$ be a hybrid system with execution $\chi$ satisfying assumptions (**A1-4**). We say the execution is *rank deficient* at a point $c_i(t_{i+1})$, $i < |\Lambda| - 1$, if $\psi_{\rho(i)}(c_i(t_{i+1}))$ does not have maximal rank, that is, if $\operatorname{rank}\left(\psi_{\rho(i)}(c_0(t_0))\right) < r$, where $r$ is the upper bound on $\operatorname{rank}(\psi_{\rho(i)})$ from Theorem 4.

The following Theorem is the main result of this paper.

*Theorem 5:* Let $\mathcal{H}$ be a hybrid system with execution $\chi$ satisfying (**A1-4**), initial condition $x_0 = c_0(t_0)$ and $i < |\Lambda| - 1$. Then $\psi_{\rho(i)}$ is rank deficient if and only if

$$\sum_{e \in E_i - \{e_{\rho(0)}\}} \dim(\operatorname{ns}(D\psi_e) \cap \operatorname{im}(D\psi_{\operatorname{sor}(e)})) > \operatorname{rank}(\psi_{e_{\rho(0)}}) - r,$$

where $r$ is the upper bound on $\psi_{\rho(i)}$ from Theorem 4.

*Proof:* The proof will follow from recursively applying the rank-nullity theorem and Lemma 2 to the sequence of linear maps (6).

First, realize that any two linear maps defined on the same domain are related by the rank-nullity theorem. In particular, it is an immediate consequence of rank-nullity that for all $j$ such that $i \geq j \geq 2$,

$$\begin{aligned}\dim(T_{c_0(t_0)}V^0) &= \operatorname{rank}(D\psi_{e_{\rho(0)}}) + \operatorname{nty}(D\psi_{e_{\rho(0)}}) \\ &= \operatorname{rank}(D\psi_{\rho(j)}) + \operatorname{nty}(D\psi_{\rho(j)}),\end{aligned}$$

where the statement is obvious for $j = 1$ since $\psi_{\rho(1)} = \phi_\tau^{\rho(1)} \circ R_{e_{\rho(0)}} = \psi_{e_{\rho(0)}}$. Thus, the rank-nullity of $\psi_{\rho(i)}$ is certainly equal to the rank-nullity of $\psi_{\rho(i-1)}$:

$$\operatorname{rank}(\psi_{\rho(i)}) + \operatorname{nty}(\psi_{\rho(i)}) = \operatorname{rank}(\psi_{\rho(i-1)}) + \operatorname{nty}(\psi_{\rho(i-1)}).$$

Applying Lemma 2 to the above equation while noting that $\psi_{\rho(i)} = \psi_{e_{\rho(i-1)}} \circ \psi_{\rho(i-1)}$ yields

$$\begin{aligned}\operatorname{rank}(\psi_{\rho(i)}) &= \operatorname{rank}(\psi_{\rho(i-1)}) \\ &\quad - \dim(\operatorname{ns}(D\psi_{e_{\rho(i-1)}}) \cap \operatorname{im}(D\psi_{\rho(i-1)})).\end{aligned}$$

If we continue in this vein by relating the rank-nullity of $\psi_{\rho(j)}$ with $\psi_{\rho(j-1)}$ for $j = i-1, \ldots, 2$, we obtain

$$\begin{aligned}\operatorname{rank}(\psi_{\rho(i)}) &= \operatorname{rank}(\psi_{e_{\rho(0)}}) \\ &\quad - \sum_{e \in E_i - \{e_{\rho(0)}\}} \dim(\operatorname{ns}(D\psi_e) \cap \operatorname{im}(D\psi_{\operatorname{sor}(e)})).\end{aligned}$$

The result follows by observing that the execution is rank deficient if and only if $r - \operatorname{rank}(\psi_{\rho(i)}) > 0$, where $r$ is the upper bound on rank from Theorem 4. ■

*Remark 4:* The left-hand side of the inequality in the statement of Theorem 5 shows that rank deficiency is primarily affected by the intersection of the nullspace of every reset map with the tangent space over the execution. To see this, realize that for any given $e \in E$, the nullspace of the edge map $\psi_e$ is the union of the tangent spaces

$$\operatorname{ns}(D\psi_e) = \left(\operatorname{ns}(D\phi_\tau^{\operatorname{tar}(e)}) \cap \operatorname{im}(DR_e)\right) \cup \operatorname{ns}(DR_e).$$

Therefore, because the nullspace of the flow to the guard on the target is small, *i.e.*, $\operatorname{nty}(D\phi_\tau^{\operatorname{tar}(e)}) = \dim(\operatorname{span}\{f_{\operatorname{tar}(e)}\}) = 1$, the nullspace of every edge map is primarily determined by $\operatorname{ns}(DR_e)$.

*Remark 5:* The right-hand side of Theorem 5 shows that the rank of the first edge map in the execution significantly affects rank deficiency. Since $r = \min_{e \in E_i}\{\operatorname{rank}(R_e)\}$, $\operatorname{rank}(\psi_{e_{\rho(0)}}) \geq r$ and the inequality will not be satisfied unless enough intersections occur in the left-hand side, or the intersections are large enough. This is a consequence of the fact that perturbations to initial conditions propagate differently through the execution depending on the starting domain $D_{\rho(0)}$.

In the next Section we discuss how to apply the above results to improve the stability properties of periodic hybrid systems, through an artifact of rank deficiency called *superstability*.

## IV. Application to Periodic Hybrid Systems

We are interested in applying the general results obtained thus far to periodic solutions of hybrid systems. To this end, we restrict our attention to hybrid systems with cyclic graphs and consider the rank properties of hybrid periodic orbits. We ultimately show that the rank of a periodic orbit is intimately related to the stability of that orbit.

*Definition 9:* A *hybrid system on a cycle* is a hybrid system $\mathcal{H} = (\Gamma, D, G, R, F)$ where $\Gamma = (Q, E)$ is a *directed cycle* such that $Q = \{q_1, \ldots, q_k\}$ is a set of $k$ vertices and $E = \{e_1 = (q_1, q_2), e_2 = (q_2, q_3), \ldots, e_k = (q_k, q_1)\} \subset Q \times Q$.

*Definition 10:* A *hybrid periodic orbit* $\mathcal{O} = (\Lambda, I, \rho, C)$ with period $T$ is an execution of the hybrid system on a cycle $\mathcal{H}$ such that for all $n \in \Lambda$,

- $\rho(n) = \rho(n + k)$,
- $I_n + T = I_{n+k}$,
- $c_n(t) = c_{n+k}(t + T)$.

*Remark 6:* Since $\mathcal{O}$ is periodic we may index the elements $S_0^n$, $S^n$, $U_0^n$, $V_0^n$ and $V^n$ defined in (**H1-2**) using the vertex set $Q$ of the graph $\Gamma$ of $\mathcal{H}$ rather than the indexing set $\Lambda$ (*e.g.*, one can take $S^n = S^{n+k}$).

*Remark 7:* As in Remark 2, we do not consider trivial hybrid systems on a cycle, so that $\mathcal{O}$ is never equivalent to the closed periodic orbit of a smooth dynamical system.

*Definition 11:* The *fundamental hybrid periodic orbit* associated with $\mathcal{O}$ is the fundamental execution $\mathcal{FO} = (\Lambda, I, \rho, C, W)$, with the fundamental matrix solutions $\Phi_n \in W$ such that $\Phi_n(t - t_n) = \Phi_{n+k}(t + T - t_{n+k})$.

Extending equations (3) and (5) to periodic orbits yields the following definition for a Poincaré map of a hybrid system.

*Definition 12:* Let $\mathcal{O}$ be a given hybrid periodic orbit of $\mathcal{H}$ with initial condition $x^* = c_0(t_0) \in D_{\rho(0)}$, where $\rho(0) =$

$q = \rho(k)$ and so $c_0(t_0) = \phi^q_\tau(c_k(t_k))$. The *hybrid Poincaré map* $P_q : V^q \to S^q$ is given by

$$P_q(x^*) = \psi_{\rho(k)} = \psi_{e_{\rho(k-1)}} \circ \cdots \circ \psi_{e_q} \quad (9)$$

It is well-known that the stability of hybrid periodic orbits is related to the stability of the hybrid Poincaré map. In particular, the following result is a corollary to Theorem 1 of [15] and the results of [5].

*Corollary 6:* Let $\mathcal{H}$ be a hybrid system with hybrid periodic orbit $\mathcal{O}$ satisfying (**A1-4**). Then $x^* = P_q(x^*)$ is an exponentially stable fixed point of the hybrid Poincaré map $P_q : V^q \to S^q$ if and only if $\mathcal{O}$ is exponentially stable.

As a discrete dynamical system, the stability of the Poincaré map is determined by the eigenvalues of its derivative evaluated at a fixed point. The following is a corollary to Theorem 4.

*Corollary 7:* The hybrid Poincaré map $P_q : V^q \to S^q$ is exponentially stable if and only if all eigenvalues of $DP_q(x^*)$ fall within the unit circle. In particular, $P_q(x^*)$ has only $r_q = \mathrm{rank}(DP_q(x^*))$ many nontrivial eigenvalues, where

$$r_q \le \min_{e\in E}\{\mathrm{rank}(R_e)\} \le \min_{q\in Q}\{\dim(D_q) - 1\},$$
$$r_q \ge \sum_{e\in E} \mathrm{rank}(R_e) - m - \sum_{q\in \mathrm{sor}(E)-\{q\}} (\dim(D_q) - 1)\,,$$

$m$ is the number of non-transverse edges in the cycle, and $E$ and $Q$ are the edge and vertex sets of $\Gamma$.

It follows that the stability of a rank deficient Poincaré map is determined by fewer eigenvalues than a Poincaré map with maximal rank. Of course, this does not imply that rank deficient orbits are more stable than orbits with maximal rank. There is, however, a specific case where rank deficiency always improves the stability properties of the Poincaré map. Recall that a discrete dynamical system that is *superstable* [6] is characterized by the derivative of the system equal to 0. When this occurs, the discrete dynamical system is said to be completely insensitive to perturbations in initial conditions. This notion adapts to periodic hybrid systems as follows.

*Definition 13:* The hybrid periodic orbit $\mathcal{O}$ with initial condition $x^*$ and its associated Poincaré map $P_q$ are said to be *superstable* at $x^*$ if $\mathrm{rank}(DP_q(x^*)) = 0$.

All eigenvalues of a superstable Poincaré map are equal to 0, implying that not only is it exponentially stable, it is completely insensitive to perturbations in initial conditions. We obtain the following Corollary to Theorem 5.

*Corollary 8:* The Poincaré map $P_q$ is superstable if and only if the lower bound on rank in Corollary 7 is equal to 0 and

$$\sum_{e\in E-\{e_q\}} \dim(\mathrm{ns}(D\psi_e) \cap \mathrm{im}(D\psi_{\mathrm{sor}(e)})) = \mathrm{rank}(\psi_{e_q}).$$

Using the Corollary and Theorem 5 it is easy to see that single-domain hybrid systems are rank deficient if and only if the reset map has rank equal to 0, in which case it is superstable. The planar compass biped, which we study in Section IV-B, is a single-domain hybrid system with a reset map that is never 0-rank, and so is never superstable. We first illustrate our results and the remarks following Theorem 5 on a simple two-domain hybrid system.

### A. Superstability

The following application is a two-domain hybrid system on a cycle $\mathcal{H} = (\Gamma, D, G, R, F)$, with graph structure $\Gamma = \big\{Q = \{1, 2\}, E = \{e_1 = (1, 2), e_2 = (2, 1)\}\big\}$, and domains $D = \{D_1, D_2\}$. We will show that this system is insensitive to perturbations to initial conditions in only one domain.

We define the first domain, $D_1$, of this system to be the upper-right quadrant of $\mathbb{R}^2$. The vector field on $D_1$ is $f_1(x, y) = \big(-y + x(1 - x^2 - y^2),\, x + y(1 - x^2 - y^2)\big)^T$. The flow in this domain is mapped to the next domain, $D_2$, when it reaches the positive $y$-axis, which is the guard on $D_1$: $G_{e_1} = \{y = 0\}$. The reset map $R_{e_1}$ is defined by the immersion of the $xy$-plane into $\mathbb{R}^3$, $R_{e_1}(x, y) = (x, y, 0)^T$, which has rank 1 on $G_{e_1}$. As an immersion, $\mathrm{nty}(DR_{e_1}) = 0$.

The second domain is the subset of $\mathbb{R}^3$ defined by $D_2 = \{x \ge 0, y \ge 0, z \ge 0\}$, with linear vector field $f_2(x, y, z)^T = (-x, -z, y)^T$, and guard $G_{e_2} = \{y = 0\}$. The flow of $f_2$ is allowed to evolve in $D_2$ until it reaches the $xz$-plane, when it is mapped back to the positive $x$-axis in $D_1$ by $R_{e_2}(x, y, z) = (x + 1, y)^T$, which has rank 1 and nullspace $\mathrm{ns}(DR_{e_2}) = \mathrm{span}\{(0, 0, 1)^T\}$.

This system has a hybrid periodic orbit $\mathcal{O}$ with an initial condition $c_0(t_0) = (1, 0)$ in $D_1$. Define the Poincaré map for initial conditions on the $x$-axis of $D_1$, $P_1 : V^1 \to S^1_0$ by $P_1 = R_{e_2} \circ \phi^2_\tau \circ R_{e_1} \circ \phi^1_\tau$ and the Poincaré map for the second domain, $P_2 : V^2 \to S^2_0$ by $P_2 = R_{e_1} \circ \phi^1_\tau \circ R_{e_2} \circ \phi^2_\tau$, where $V^2$ is the positive $y$ axis of $D_2$. It is easy to see that $R_{e_1}$ is transverse to $f_2$ and $R_{e_2}$ is transverse to $f_1$; thus the image of both reset maps is transverse to the flow on the target domain. With no non-transverse edges, Corollary 7 implies that $0 \le \mathrm{rank}(P_1) \le 1$, $0 \le \mathrm{rank}(P_2) \le 1$. Since the maximum rank of both maps is 1, in this case rank deficiency would also imply superstability.

Applying Theorem 5 directly, we see that in order for $P_2$ to be rank deficient the following inequality must be true,

$$\mathrm{ns}(DR_{e_1}) \cap \mathrm{im}\left(D\phi^1_\tau \circ DR_{e_2} \circ D\phi^2_\tau\right) > \mathrm{rank}(R_{e_2} \circ \phi^2_\tau) - 1,$$

where the right-hand side evaluates to 0. However, as noted above, $DR_{e_1}$ is an immersion and has no nullspace, so the left-hand side also evaluates to 0 and $P_2$ cannot possibly be rank deficient. In order for $P_1$ to be rank deficient,

$$\mathrm{ns}(DR_{e_2}) \cap \mathrm{im}\left(D\phi^2_\tau \circ DR_{e_1} \circ D\phi^1_\tau\right) > 0$$

must be true. Since we can find exact expressions for the flows of both $\phi^1_\tau$ and $\phi^2_\tau$, we obtain $D\phi^1_\tau$ and $D\phi^2_\tau$ by taking partial derivatives, yielding

$$\mathrm{im}(D\phi^2_\tau \circ DR_{e_1} \circ D\phi^1_\tau) = \mathrm{span}\{(0, 0, 1)^T\},$$

which, as noted above, is $\mathrm{ns}(DR_{e_2})$. Thus, the nullspace of the reset map aligns with the tangent space over the execution and $P_2$ is rank deficient. Applying Corollary 8, we confirm that the superstability condition is satisfied for $P_1$:

$$\dim(\mathrm{span}\{(0, 0, 1)^T\}) = 1 = \mathrm{rank}(R_{e_1} \circ \phi^1_\tau).$$

The superstable behavior is confirmed in Figure 1, which shows initial conditions in $D_1$ converging to the limit cycle after one iteration.

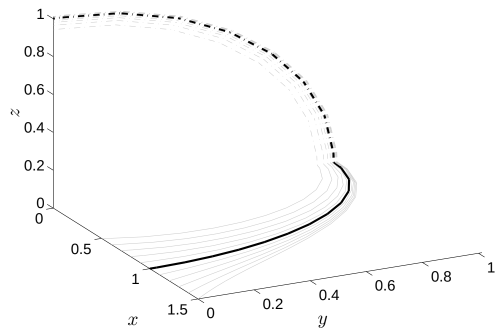


Fig. 1. Limit cycle of the superstable two-domain system. All trajectories with initial conditions in the $xy$ plane (gray) converge to the superstable cycle (black) after 1 complete traversal of the cycle.

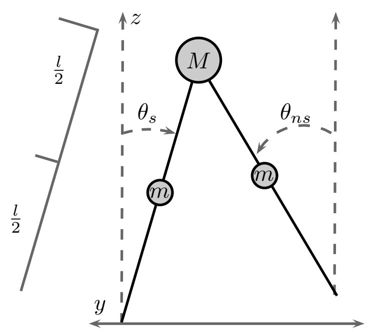


Fig. 2. Compass biped dimensions, point-mass locations and measuring conventions.

The rank deficiency and superstability analysis of this application was aided by the simplicity of the vector fields in each domain. In general it is necessary to integrate the fundamental matrix solutions on each domain in order to determine the rank deficiency and superstability properties of more complicated systems.

### B. Planar compass biped

In this subsection we consider the compass biped, a single-domain periodic hybrid system that has been studied extensively in multiple contexts, with some recent work including [7] and [16]. The objective of this application is to show that the eigenvalues of the fundamental execution accurately predict period-doubling bifurcations.

Period-doubling bifurcations of hybrid systems have been studied extensively. In [17], [18] the authors study bifurcations using formula that are not well-defined for arbitrary, multi-domain hybrid systems [5]. In [19] the bifurcation analysis is aided by the fact that the Poincaré maps are known exactly. The dynamics of bipedal systems, however, are nonlinear and sufficiently complicated as to require numerical integration for complete analysis of either their bifurcation or rank deficiency properties. The equations and analysis described in previous Sections allow for this analysis.

The compass biped is a 2-link planar robotic mechanism capable of walking down a shallow slope without control. The links form the biped's legs, with the rotary joint connecting them forming its hip. The stance link is assumed fixed to the slope and the nonstance link is free to swing above the slope. The hybrid system model for this simple mechanism is $\mathcal{H} = (\Gamma, D, G, R, F)$ with graph structure $\Gamma = \{Q = \{q\}, E = \{e_q = (q, q)\}\}$. As this is a 2-link mechanism, the dynamics evolve on the tangent bundle to the configuration space $\Theta := \mathbb{T}^2$. We give the dynamics on $D_q$ coordinates $\theta = (\theta_s, \theta_{ns}, \dot{\theta}_s, \dot{\theta}_{ns})^T$, where the angles of the stance and nonstance legs from the vertical are denoted $\theta_s$ and $\theta_{ns}$, respectively. We denote the vector field describing the biped dynamics as $\dot{\theta} = f_q(\theta; p)$, where $p$ is the angle of the slope from the horizontal. The notation $f_q(\theta; p)$ indicates that $\theta$ is an argument of the function $f_q$ and $p$ is a parameter. The guard, $G_e$, is defined by the holonomic constraint function $h : D_q \to \mathbb{R}$ corresponding to the shallow slope, $h(\theta) = (\sin(\theta_s) - \sin(\theta_{ns})) \tan(p) + (\cos(\theta_s) - \cos(\theta_{ns}))$ such that $G_e = \{h(\theta) = 0\}$. When the nonstance leg impacts the slope we model the jump in link velocities as an instantaneous plastic impact using the reset map $R_e : G_e \to R_e(G_e)$. We refer the reader interested in further modeling details to [20] and [21] for a comprehensive overview.

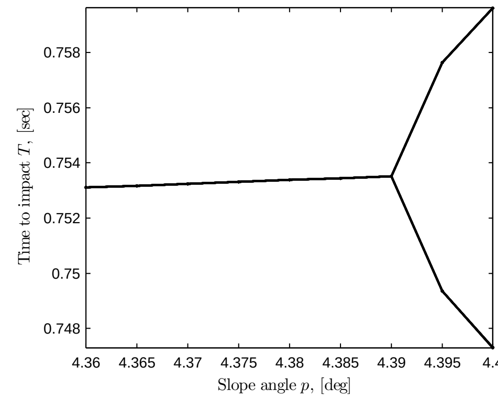


Fig. 3. Time to impact the ground versus the slope angle. After the slope increases beyond $4.39°$, two impacts are necessary for the biped to complete a single hybrid periodic orbit: the biped "limps" down the slope.

Let $\mathcal{O}$ be a hybrid periodic orbit for $\mathcal{H}$ with fixed point $\theta^*$ at the first period-doubling bifurcation, denoted $p^*$. Define the Poincaré map, $P_q : V^q \to S^q$, as in (9). Because $\mathcal{H}$ has only one domain, and $R_e$ has full rank on the guard Corollary 7 implies that $\text{rank}(P_q) = \text{rank}(R_e) = 3$.

The derivative of the reset map is obtained by simply taking partial derivatives with respect to $\theta$. The total derivative of the flow, $D\phi^q_\tau(R_e(\theta^*))$, is obtained from (4) with $x_1 = P_q(\theta^*)$, $h_j = h$ the function defining the guard, and $f_{\rho(j)}(x_1) = f_q(P_q(\theta^*); p^*)$.

It is well-known [6] that the linearization of a discrete dynamical system has an eigenvalue equal to $-1$ at a period-doubling bifurcation. By varying the ground slope we find that the eigenvalues of $DP_q(\theta^*)$ cross $-1$ at $p^* = 4.39°$ (see Figure 4):

$$\sigma(DP_q(\theta^*)) = (-0.9990, 0.1056, 1.3846\text{E-}15, -0.3053),$$
$$\theta^* = (0.385171, -0.231931, 1.729380, 2.183038)^T.$$

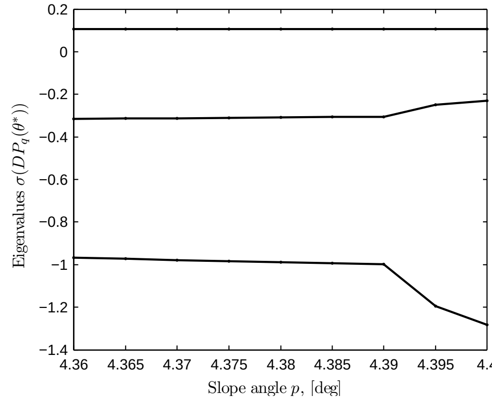


Fig. 4. The eigenvalues of $DP_q(\theta^*)$ versus slope angle. One eigenvalue crosses $-1$ at $p = 4.39°$.

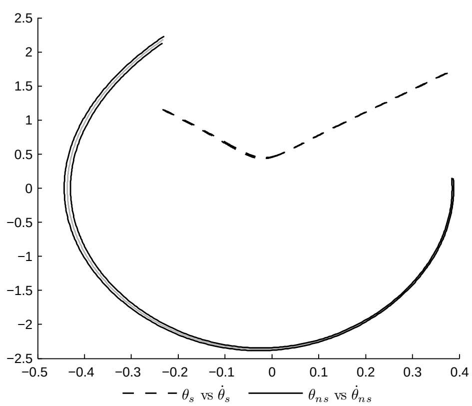


Fig. 5. Phase portrait for 2 steps of the compass biped, with $p = 3.485°$ (gray) for just before and $p = 3.495°$ (black) after the bifurcation. The non-stance leg (solid) exhibits the "limping" behavior after bifurcation.

Figure 3, which plots the time $T$ it takes for the swing leg to impact the ground over several steps, confirms that we do indeed have a period-doubling bifurcation at $p^* = 3.95°$, and so we have confirmed that the eigenvalues of our Poincaré map correctly predict the bifurcation behavior of the system.

## V. Conclusion

The results presented in this paper emphasize fundamental differences between smooth smooth and hybrid systems, implying a depth to hybrid systems that is not yet fully understood. We have shown that the rank of a hybrid execution is always less than the dimension of the space on which solutions evolve. The upper and lower bounds on the rank are known *a priori*. The rank deficiency condition is determined by the alignment of the nullspace of each reset map with the tangent space to the execution. We applied our results to a periodic orbit and observed superstability, which is a desirable artifact of rank deficiency, and noted that the linearization of the Poincaré map predicts the correct eigenvalue behavior of the system.

A future research direction is to take advantage of existing tools — such as those in [22] — and design controllers for hybrid systems that directly induce rank deficiency and superstability, and hence improved robustness to perturbations.

## References


[1] J. Lygeros, K. Johansson, S. Sastry, and M. Egerstedt, "On the existence of executions of hybrid automata," in *Decision and Control, 1999. Proceedings of the 38th IEEE Conference on*, vol. 3, 1999, pp. 2249 –2254 vol.3.

[2] J. Lygeros, K. Johansson, S. Simic, J. Zhang, and S. Sastry, "Dynamical properties of hybrid automata," *Automatic Control, IEEE Transactions on*, vol. 48, no. 1, pp. 2 – 17, Jan. 2003.

[3] G. Davrazos and N. Koussoulas, "A review of stability results for switched and hybrid systems," in *Mediterranean Conference on Control and Automation*, 2001.

[4] A. Lamperski and A. Ames, "Lyapunov-like conditions for the existence of zeno behavior in hybrid and lagrangian hybrid systems," in *Decision and Control, 2007 46th IEEE Conference on*, 2007, pp. 115 –120.

[5] E. D. Wendel and A. D. Ames, "Rank Properties of Poincaré Maps for Hybrid Systems with Applications to Bipedal Walking," in *HSCC '10: Proceedings of the 13th ACM international conference on Hybrid systems: computation and control*. ACM, 2010, pp. 151–160.

[6] S. H. Strogatz, *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering*. Cambridge, MA: Westview Press, 2000.

[7] J. Grizzle, G. Abba, and F. Plestan, "Asymptotically stable walking for biped robots: Analysis via systems with impulse effects," *IEEE Transactions on Automatic Control*, vol. 46, no. 1, pp. 51–64, 2001.

[8] M. Hirsch and S. Smale, *Differential equations, dynamical systems, and linear algebra*. Academic Press Inc, 1974.

[9] L. Perko, *Differential equations and dynamical systems*. Springer, 2001.

[10] T. Parker and L. Chua, *Practical numerical algorithms for chaotic systems*. Springer-Verlag New York, Inc. New York, NY, USA, 1989.

[11] H. Khalil, *Nonlinear systems*. Prentice hall Englewood Cliffs, NJ, 1996.

[12] S. Smale, "Stable manifolds for differential equations and diffeomorphisms," *Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser.*, vol. 17, pp. 97–116, 1963.

[13] S. Lang, *Introduction to linear algebra*, 2nd ed. New York: Springer-Verlag, 1986.

[14] E. Wendel and A. Ames, "Rank deficiency and superstability of hybrid systems," *submitted to Nonlinear Analysis: Hybrid Systems*, 2011. [Online]. Available: people.tamu.edu/ ericdbw/NAHS2011.pdf

[15] B. Morris and J. Grizzle, "A restricted poincare map for determining exponentially stable periodic orbits in systems with impulse effects: Application to bipedal robots," in *44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05*, 2005, pp. 4199–4206.

[16] D. Pekarek, A. Ames, and J. Marsden, "Discrete mechanics and optimal control applied to the compass gait biped," in *IEEE Conference on Decision and Control and European Control Conference ECC, New Orleans, USA*, 2007.

[17] I. Hiskens, "Stability of hybrid system limit cycles: Application to the compass gait biped robot," in *IEEE Conference on Decision and Control*, vol. 1. IEEE; 1998, 2001, pp. 774–779.

[18] R. Leine and H. Nijmeijer, *Dynamics and bifurcations of non-smooth mechanical systems*. Springer Verlag, 2004.

[19] M. di Bernardo, K. Johansson, and F. Vasca, "Self-oscillations and sliding in relay feedback systems: Symmetry and bifurcations," *International Journal of Bifurcations and Chaos*, vol. 11, no. 4, pp. 1121–1140, 2001.

[20] E. R. Westervelt, *Feedback Control of Dynamic Bipedal Robot Locomotion*, ser. Control and automation. Boca Raton: CRC Press, 2007, no. 1.

[21] A. Goswami, B. Thuilot, and B. Espiau, "Compass-like biped robot part i: Stability and bifurcation of passive gaits," INRIA, Tech. Rep. 2996, October 1996.

[22] D. Tyner and A. Lewis, "Jacobian linearisation in a geometric setting," in *Decision and Control, 2003. Proceedings. 42nd IEEE Conference on*, vol. 6, 2003, pp. 6084 – 6089 Vol.6.