Control Barrier Function based Quadratic Programs with Application to Bipedal Robotic Walking

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Abstract— **This paper presents a methodology for the development of control barrier functions (CBFs) through a backstepping inspired approach. Given a set defined as the superlevel set of a function, h, the main result is a constructive means for generating control barrier functions that guarantee forward invariance of this set. In particular, if the function defining the set has relative degree n, an iterative methodology utilizing higher order derivatives of h provably results in a control barrier function that can be explicitly derived. To demonstrate these formal results, they are applied in the context of bipedal robotic walking. Physical constraints, e.g., joint limits, are represented by control barrier functions and unified with control objectives expressed through control Lyapunov functions (CLFs) via quadratic program (QP) based controllers. The end result is the generation of stable walking satisfying physical realizability constraints for a model of the bipedal robot AMBER2.**

I. Introduction

Humans can perform many difficult dynamic behaviors with ease, including: crawling, climbing and — of special focus in this paper — walking. The core of performing these tasks is the ability to satisfy structural and physical constraints while simultaneously realizing dynamics based control objectives. Realizing this balance between safety constraints and control objectives in the context of dynamic behaviors has yet to be fully realized on robotic systems. A core issue preventing this is the unification of safety and control objectives in a single unified framework—one that can be solved online in real-time, i.e., does not require *a priori* optimization, while still yielding formal guarantees of correctness. The goal of this paper is to present a methodology for realizing physical constraints on robotic systems through control barrier functions, and balancing these constraints through control objectives represented as control Lyapunov functions expressed through a unified quadratic program based control methodology. The application of these ideas to robotic walking will demonstrate their affectiveness in ensuring physical constraints during dynamic behaviors.

Barrier functions, first utilized in numerical optimization methods [7], [23], are continuous functions whose values approach infinity when the state approaches the boundary of a set. For instance, given a set $\mathcal{C}$, $B(x)$ is a barrier function, if $B(x) \to \infty$ as $x \to \partial\mathcal{C}$. The concept recently has been related to control Lyapunov functions for the purposes of constructing nonlinear controllers [22]. In particular, Lyapunov-like barrier functions have been established; that is, employing derivatives of barrier functions guarantees the invariance of set $\mathcal{C}$, e.g. $\dot{B}(x,u) < 0$ [21], [19]. With a view towards expanding the class of control inputs that imply set invariance, recent work has focused on a new class of barrier functions that ensure set invariance while yielding a larger set of control inputs [4]. In particular, if $\mathcal{C}$ is the superlevel set of a function $h(x)$, there is the corresponding barrier function candidate:

$$B(x) = -\log\left(\frac{h(x)}{1+h(x)}\right) \tag{1}$$

which is a valid barrier function if it satisfies the condition:

$$\dot{B}(x,u) < \frac{\gamma}{B(x)} \tag{2}$$

for $\gamma \geq 0$. Importantly, this allows $B(x)$ to grow when it is far away from the boundary of $\mathcal{C}$ while still provably yielding set invariance [4]. Necessary to $B(x)$ being a valid barrier is its satisfaction of (2); yet there may not exist control inputs that satisfy this condition (for example, if the relative degree [11] of h is ≥ 2). The objective of this paper is to provide quantifiable conditions for the existence of such control inputs through the use of relative degree conditions on h coupled with backstepping inspired methodology.

Motivated by the use of backstepping methods for Lyapunov functions, as first developed by Kokotović [13], [14], the main result of this paper is a methodology for constructing barrier functions that are guaranteed to satisfy the condition (2). Beginning with a barrier function candidate of the form (1) and assuming that the function h that defines the set $\mathcal{C}$ has relative degree n, we are able to expand the barrier function candidate through "virtual" inputs that satisfy (2) and, through iteration on these virtual inputs, ultimately yield the "true" control input that satisfies (2). The end result is a barrier function, dependent on h and its higher order derivatives, that provably satisfies the barrier function conditions. This is formally established through the main result of the paper.

To apply the main result of the paper to bipedal robotic walking, it is necessary to unify control barrier functions (CBFs)—which encode safety and physical constraints—with control objectives necessary to achieve locomotion as encoded by control Lyapunov functions. Control Lyapunov functions (CLFs), as pioneered by Artstein and Sontag [6], [20], have been widely used in nonlinear control [16], [8]. In the context of bipedal robotic locomotion, control Lyapunov

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functions have been utilized in the context of quadratic programs (QPs) to balance control objectives with torque based constraints, with the results being applied experimentally [5], [3], [9]. This sets the stage for the main application of the results of this paper: as motivated by [4], a CBF-CLF QP is introduced in the context of bipedal locomotion that ensures safety through control barrier functions and control objectives expressed as control Lyapunov functions. This formulation is applied to a hybrid system model of the bipedal robot AMBER2, with the end result being stable walking in simulation that satisfies physical constraints encoded as control barrier functions.

II. Control Barrier Functions via Backstepping Inspired Methods

This section develops and presents the main formal results of the paper: a backstepping inspired methodology for constructing control barrier functions (CBFs). We begin by introducing the form of barrier functions considered in this paper, as introduced in [4], defined for a set $\mathcal{C}$ that is the super level set of a function $h(x)$. Motivated by the use of backstepping in generating Lyapunov functions [21], [13], we assume that h has relative degree n and utilize the higher order derivative of h to iteratively construct valid control barrier functions. The end product of this procedure yields the main result of this paper: a formal guarantee that the resulting control barrier function is valid, i.e., that control inputs exist that satisfy the barrier function condition (2). With a view toward the application of this method to robotic walking, we conclude this section by briefly reviewing control Lyapunov functions (CLFs), and the unification of CLFs and CBFs through quadratic program (QP) based controllers.

A. Control Barrier Functions

Consider an affine control system:

$$\dot{x} = f(x) + g(x)u, \tag{3}$$

for $x \in \mathbb{R}^n$ and $u \in U = \mathbb{R}^m$ with f and g assumed to be locally Lipschitz. Given a set $\mathcal{C} \subset \mathbb{R}^n$, we determine conditions on functions $B : \mathcal{C} \to \mathbb{R}$ such that solutions to (3), with initial condition in $\mathcal{C}$, remain in $\mathcal{C}$ for all time. First, since (3) is assumed to be locally Lipschitz for any initial condition $x_0 \in \mathbb{R}^n$, there exists a maximum time interval $I(x_0) = [0, \tau_{\max})$ such that $x(t)$ is the unique solution to (3) on $I(x_0)$; in the case when f is forward complete, $\tau_{\max} = \infty$. The set $\mathcal{C}$ is *forward invariant* if for every $x \in \mathcal{C}$, $x(t) \in \mathcal{C}$ for all $t \in I(x)$.

Consider the set $\mathcal{C}$ defined by

$$\mathcal{C} = \{x \in \mathbb{R}^n : h(x) \geq 0\}, \tag{4}$$

$$\partial\mathcal{C} = \{x \in \mathbb{R}^n : h(x) = 0\}, \tag{5}$$

$$\text{Int}(\mathcal{C}) = \{x \in \mathbb{R}^n : h(x) > 0\}. \tag{6}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function.

With a view toward ensuring forward invariance of $\mathcal{C}$, we consider the following definition [4]:

Definition 1. *Let $\mathcal{C} \subset \mathbb{R}^n$ be defined by* (4)-(6) *with $h : \mathbb{R}^n \to \mathbb{R}$ continuously differentiable, then a function $B : \mathcal{C} \to \mathbb{R}$ is a* ***control barrier function (CBF)*** *if there exist class $\mathcal{K}$ functions α_1, α_2 and a constant $\gamma > 0$ such that*

$$\frac{1}{\alpha_1(h(x))} \leq B(x) \leq \frac{1}{\alpha_2(h(x))}, \tag{7}$$

$$\inf_{u \in U} [L_f B(x) + L_g B(x) u - \frac{\gamma}{B(x)}] \leq 0. \tag{8}$$

Using the definition of a CBF, we can consider the set consisting of all control inputs that guarantee the forward invariance of set $\mathcal{C}$:

$$K_B(x) = \quad \{u \in U : L_f B(x) + L_g B(x) u - \frac{\gamma}{B(x)} \leq 0\}.$$

This yields the following result from [4]:

Theorem 1. *Given a set $\mathcal{C} \subset \mathbb{R}^n$ defined by* (4)-(6) *with associated control barrier function B, any Lipschitz continuous controller $u(x) \in K_B(x)$ for the system* (3) *renders the set $\mathcal{C}$ forward invariant.*

B. Main Result: CBFs via Backstepping

The main observation that motivates our result is: $K_B(x)$ may be empty when $L_g B = 0$. More specifically, if $h(x)$ has relative degree greater than 1, then $L_g B = 0$; if furthermore $L_f B(x) \geq \frac{\gamma}{B(x)}$, then $K_B(x)$ is empty and therefore no control can make (8) satisfied. However, physical constraints on robotic systems, as represented by sets $\mathcal{C}$, will be guaranteed to have relative degree greater than 1 if the constraints only depend on the configuration of the robot. As a result, it is necessary to find a methodology for dealing with this situation. Motivated by [21], we introduce a backstepping inspired methodology for the form of control barrier functions considered in this paper.[1] The control barrier functions calculated through our methods render the $\mathcal{C}$ forward invariant when $h(x)$ has a relative degree great than 1.

To make the inputs appear in derivatives of the control barrier function, $h(x)$ will be utilized in the context of dynamic extension. Suppose $h(x)$ has (vector) relative degree 2, wherein it follows that: :

$$\phi_1(x) = h(x), \tag{9}$$

$$\dot{\phi}_1(x) = \phi_2(x), \tag{10}$$

$$\dot{\phi}_2(x, \dot{x}) = L_f L_f h(x) + L_g L_f h(x) u, \tag{11}$$

and a set $\mathcal{C}$ is defined by (4)-(6). We can pick a function $B_1(x)$ so that $B_1(x)$ satisfies the inequality

$$\frac{1}{\alpha_{1,1}(h(x))} \leq B_1(x) \leq \frac{1}{\alpha_{1,2}(h(x))}, \tag{12}$$

where $\alpha_{1,1}$ and $\alpha_{1,2}$ are class $\mathcal{K}$ functions. Note that an example of such a candidate is given by

$$B_1(x) = -\log\left(\frac{h(x)}{1 + h(x)}\right). \tag{13}$$

[1]Note that the form of barrier functions considered in this paper is very different from that in [21]; the end result is a more general methodology for constructing control barrier functions.

In addition, let $h_1(x) = \phi_2(x) - \xi_1$, where ξ_1 is a stabilizing function we have to design. The time derivative of $B_1(x)$ is thus given by:

$$\dot{B}_1 = \frac{dB_1(x)}{dh}\dot{h} = \frac{dB_1(x)}{dh}\phi_2 = \frac{dB_1(x)}{dh}(h_1(x) + \xi_1).$$

Picking $\xi_1 = 0$ results in:

$$\dot{B}_1 = \frac{dB_1(x)}{dh}h_1(x) = \frac{dB_1(x)}{dh}\dot{h}.$$

Motivated by the backingstepping method for Lyapunov functions [12], we can define a *control barrier function candidate* with $\dot{h}$:

$$B_2(x) := B_1(x) + E(\dot{h}(x)), \tag{14}$$

where $E(\dot{h}(x))$ has the following properties:

$$\inf_{x \in \text{Int}(\mathcal{C})} E(\dot{h}(x)) \geq 0, \tag{15}$$

$$\sup_{x \in \text{Int}(\mathcal{C})} E(\dot{h}(x)) \leq E_{max}, \tag{16}$$

$$\frac{dE(\dot{h}(x))}{d\dot{h}(x)} = 0 \quad \text{if and only if} \quad \dot{h}(x) = 0, \tag{17}$$

where E_{max} is a positive constant depending on the choice of $E(\dot{h}(x))$. An concrete example of a function satisfying this properties is given by:

$$E(\dot{h}(x)) = a_E \frac{b_E \dot{h}(x)^2}{1 + b_E \dot{h}(x)^2} \tag{18}$$

where a_E and b_E are the positive parameters that can be chosen. These allow us to state the main result of the paper.

Theorem 2. *Given a set $\mathcal{C} \subset \mathbb{R}^n$ defined by* (4)-(6)*, if $h(x)$ has relative degree 2, then $B_2(x)$ given in* (14) *is a control barrier function and any Lipschitz continuous controller $u(x) \in K_{B_2}(x)$ renders the set $\mathcal{C}$ forward invariant.*

Before proving the theorem, the following Lemma must be established.

Lemma 3. *Given a set $\mathcal{C} \subset \mathbb{R}^n$ defined by* (4)-(6)*, if a function $B(x) : \mathcal{C} \to \mathbb{R}$ for a continuously differentiable function $h(x) : \mathbb{R}^n \to \mathbb{R}$ satisfies the following conditions:*

$$\inf_{x \in \text{Int}(\mathcal{C})} B(x) \geq 0, \quad \lim_{x \to \partial\mathcal{C}} B(x) = \infty,$$

and $B(x) \to \infty$ if and only if $x \to \partial\mathcal{C}$, then there exist class $\mathcal{K}$ functions α_1, α_2 such that

$$\frac{1}{\alpha_1(h(x))} \leq B(x) \leq \frac{1}{\alpha_2(h(x))}.$$

Proof. Taking the reciprocal of $B(x)$, we have

$$\inf_{x \in \text{Int}(\mathcal{C})} \frac{1}{B(x)} \geq 0, \quad \lim_{x \to \partial\mathcal{C}} \frac{1}{B(x)} = 0.$$

Define a function $\psi(s)$ as

$$\psi(s) = \inf_{\{x | s \leq h(x)\}} \frac{1}{B(x)}, \quad \forall s \geq 0.$$

Because $\frac{1}{B(x)}$ approaches zero if and only if $h(x)$ is close to zero, the function $\psi(s)$ is continuous, positive definite, and increasing but not necessarily strictly increasing. Therefore, a class $\mathcal{K}$ function $\alpha_2(s)$ can be established such that $\alpha_2(s) \leq \psi(s)$, that is,

$$\frac{1}{B(x)} \geq \psi(h(x)) \geq \alpha_2(h(x)). \tag{19}$$

Similarly, define a function $\phi(s)$ by

$$\phi(s) = \sup_{\{x | 0 \leq h(x) \leq s\}} \frac{1}{B(x)}, \quad \forall s \geq 0.$$

The function $\phi(s)$ thus is continuous, positive definite and increasing but not necessarily strictly increasing. Therefore, a class $\mathcal{K}$ function $\alpha_1(s)$ can be established such that $\alpha_1(s) \geq \phi(s)$, that is,

$$\frac{1}{B(x)} \leq \phi(h(x)) \leq \alpha_1(h(x)). \tag{20}$$

Taking the reciprocals of (19) and (20) we have

$$\frac{1}{\alpha_1(h(x))} \leq B(x) \leq \frac{1}{\alpha_2(h(x))}.$$

□

Now, we can apply this Lemma to the proof of Theorem 2.

Proof. By assumption, and specifically (12), B_1 has the properties

$$\inf_{x \in \text{Int}(\mathcal{C})} B_1(x) \geq 0, \quad \lim_{x \to \partial\mathcal{C}} B_1(x) = \infty.$$

By (14), we know that

$$\inf_{x \in \text{Int}(\mathcal{C})} B_2(x) \geq 0, \quad \lim_{x \to \partial\mathcal{C}} B_2(x) = \infty,$$

because $E(\dot{h})$ is always positive and bounded. It is also known that $B_2(x) \to \infty$ if and only if $x \to \partial\mathcal{C}$, so B_2 satisfies the following condition by Lemma 3:

$$\frac{1}{\alpha_{2,1}(h(x))} \leq B_2(x) \leq \frac{1}{\alpha_{2,2}(h(x))}, \tag{21}$$

where $\alpha_{2,1}$ and $\alpha_{2,2}$ are class $\mathcal{K}$ functions, which implies that (7) is satisfied.

To establish (8), taking the time derivative of B_2 yields:

$$\begin{aligned}\dot{B}_2 &= \dot{B}_1 + \frac{dE(\dot{h})}{d\dot{h}}\ddot{h} \\ &= \frac{dB_1(x)}{dh}\dot{h} + \frac{dE(\dot{h})}{d\dot{h}}(L_f L_f h(x) + L_g L_f h(x) u).\end{aligned}$$

Therefore, (8) yields the following condition:

$$\frac{dB_1(x)}{dh}\dot{h} + \frac{dE(\dot{h})}{d\dot{h}}(L_f L_f h(x) + L_g L_f h(x) u) \leq \frac{\gamma}{B_2(x)}. \tag{22}$$

If $\frac{dE(\dot{h})}{d\dot{h}} = 0$, then $\dot{h} = 0$ and thus the left hand side of (22) equals to 0, which means that (22) holds. If $\frac{dE(\dot{h})}{d\dot{h}} \neq 0$, rearranging (22), we obtain the final inequality constraint:

$$u \leq \frac{1}{\frac{dE(\dot{h})}{d\dot{h}} L_g L_f h(x)} \left(\frac{\gamma}{B_2(x)} - \frac{dB_1(x)}{dh}\dot{h} \right) - \frac{L_f L_f h(x)}{L_g L_f h(x)}, \tag{23}$$

if $\frac{dE(\dot{h})}{d\dot{h}} L_g L_f h(x) > 0$, and

$$u \geq \frac{1}{\frac{dE(\dot{h})}{d\dot{h}} L_g L_f h(x)} \left(\frac{\gamma}{B_2(x)} - \frac{dB_1(x)}{dh}\dot{h} \right) - \frac{L_f L_f h(x)}{L_g L_f h(x)}, \tag{24}$$

if $\frac{dE(\dot{h})}{d\dot{h}} L_g L_f h(x) < 0$, where $L_g L_f h(x)$ is non-singular because of the assumption that $h(x)$ has relative degree 2. As a result, an input u satisfying (23) and (24) are guaranteed to exist. According to Definition 1, B_2 is a *control barrier function* because (21) and (22) hold. Finally, from Theorem 1, any Lipschitz continuous control input u satisfying (22) renders the set $\mathcal{C}$ forward invariant.

□

Having established Theorem 2, the method can be extended to the case where $h(x)$ has relative degree n where $n \geq 2$. Consider the coordinates:

$$\begin{aligned} \phi_1(x) &= h(x), \\ \dot{\phi}_1(x) &= \phi_2(x), \\ \dot{\phi}_2(x) &= \phi_3(x), \\ &\vdots \\ \dot{\phi}_n(x, \dot{x}) &= L_f^n h(x) + L_g L_f^{n-1} h(x) u, \end{aligned} \tag{25}$$

Following the backstepping methodology, and motivated by the control barrier function considered in (14), define the following control barrier function candidate:

$$B_n(x) = B_1(x) + \sum_{i=1}^{n-1} E_i(h_i), \tag{26}$$

where $h_i = \phi_{i+1} - \xi_i$, $\xi_1 = 0$, $\xi_2 = \frac{-1}{\frac{dE_1}{dh_1}} \frac{dB_1}{dh} h_1$, $\xi_i = \dot{\xi}_{i-1} - \frac{1}{\frac{dE_{i-1}}{dh_{i-1}}} \frac{dE_{i-2}}{dh_{i-2}} h_{i-1}$ and $E_i(h_i)$ has the properties given in (15) - (17).

The derivative of $B_n(x)$ thus can be calculated as

$$\begin{aligned} \dot{B}_n(x) =& \frac{dE_{n-2}}{dh_{n-2}} h_{n-1} \\ &+ \frac{dE_{n-1}}{dh_{n-1}} \left(L_f^n h(x) + L_g L_f^{n-1} h(x) u - \dot{\xi}_{n-1} \right). \end{aligned}$$

It is easy to check that the above equation still holds if $\frac{dE_i}{dh_i} = 0$ for any $1 \leq i \leq n-1$. Finally, if $\frac{dE_{n-1}}{dh_{n-1}} = 0$, then $\dot{B}_n(x) = 0$, which is guaranteed to be smaller than $\frac{\gamma}{B_n(x)}$; if $\frac{dE_{n-1}}{dh_{n-1}} \neq 0$, the control law can be determined through the inequality:

$$\begin{aligned} u \leq& \frac{1}{\frac{dE_{n-1}}{dh_{n-1}} L_g L_f^{n-1} h(x)} \left(\frac{\gamma}{B_n(x)} - \frac{dE_{n-2}}{dh_{n-2}} h_{n-1} \right) \\ &- \frac{L_f^n h(x) - \dot{\xi}_{n-1}}{L_g L_f^{n-1} h(x)}, \end{aligned}$$

if $\frac{dE_{n-1}}{dh_{n-1}} L_g L_f^{n-1} h(x) > 0$, and

$$\begin{aligned} u \geq& \frac{1}{\frac{dE_{n-1}}{dh_{n-1}} L_g L_f^{n-1} h(x)} \left(\frac{\gamma}{B_n(x)} - \frac{dE_{n-2}}{dh_{n-2}} h_{n-1} \right) \\ &- \frac{L_f^n h(x) - \dot{\xi}_{n-1}}{L_g L_f^{n-1} h(x)}, \end{aligned}$$

if $\frac{dE_{n-1}}{dh_{n-1}} L_g L_f^{n-1} h(x) < 0$, where $L_g L_f^{n-1} h(x)$ is non-singular because $h(x)$ has relative degree n by assumption. Therefore, u is guaranteed to exist.

Using the same argument as in Theorem 2, it is easy to show that for a relative degree n output $h(x)$, the function $B_n(x)$ defined by (26) is a CBF. This is summarized in the following theorem.

Theorem 4. *Given a set $\mathcal{C} \subset \mathbb{R}^n$ defined by* (4)-(6)*, if $h(x)$ has relative degree n, then $B_n(x)$ as given in* (26) *is a control barrier function and any Lipschitz continuous controller $u(x) \in K_{B_n}(x)$ renders the set $\mathcal{C}$ forward invariant.*

C. Control Lyapunov Functions

Consider the affine nonlinear control system (3) where it is assumed that $f(0) = 0$. To achieve exponential stability of x to 0, we utilize a special class of control Lyapunov functions $V(x)$ termed *rapidly exponentially stabilizing control Lyapunov function (RES-CLF)* [3]. This yields the set of control inputs that exponentially stabilizes the system (3):

$$K_\varepsilon(x) = \{u \in U : L_f V_\varepsilon(x) + L_g V_\varepsilon(x) u + c_3 V_\varepsilon(x) \leq 0\}.$$

D. Combining CLFs and CBFs via QPs

To unify CLFs and CBFs, we can formulate a QP as

$$u^*(x) = \underset{\mathbf{u}=\begin{bmatrix} u \\ \delta \end{bmatrix} \in \mathbb{R}^{m+1}}{\text{argmin}} \quad \frac{1}{2}\mathbf{u}^T H(x) \mathbf{u} + F(x)^T \mathbf{u} \tag{27}$$

$$\begin{aligned} s.t. \quad & \psi_0(x) + \psi_1^T(x) u \leq \delta, && \text{(CLF)} \\ & L_f B(x) + L_g B(x) u \leq \frac{\gamma}{B(x)}, && \text{(CBF)} \end{aligned}$$

where $\psi_0(x) = L_f V(x) + c_3 V(x)$, $\psi_1(x) = L_g V(x)^T$, $H(x) \in \mathbb{R}^{m+1 \times m+1}$ and $F(x) \in \mathbb{R}^{m+1}$ are arbitrarily smooth cost functions that can be chosen based upon desired (state based) weighting of the control inputs, and $\delta \in \mathbb{R}$ is a relaxation of the CLF constraint, which can be chosen to guarantee a feasible solution to the QP. Coupling the results from [4], [18] and Theorem 4, we have the following theorem.

Theorem 5. *Given a set $\mathcal{C} \subseteq \mathbb{R}^n$ defined by* (4)-(6) *with h relative degree n and $B = B_n$ an associated control barrier function given in* (26)*, any Lipschitz continuous control law $u^*(x)$ obtained by solving the QP* (27) *renders the set $\mathcal{C}$ forward invariant.*

In practice, if $x \in \text{Int}(\mathcal{C})$ are far away from the boundary, $\partial\mathcal{C}$, the control objective (as represented by the CLF) will be achieved exponentially; otherwise, it will be violated depending on how x close to $\partial\mathcal{C}$ (as dictated by the CBF).

III. Robotic Modeling and Control

The main goal of this paper is to apply control barrier functions to the control of bipedal walking robots. In this section, based on the mathematical model of the 7-link bipedal robot, AMBER2 [17], we review the construction of output functions that are utilized to formulate control Lyapunov functions. Then, we present a multi-objective QP based controller representing a combination of control objectives, torque and forced based constraints. This will set the stage for the utilization of control barrier functions in the context of bipedal robotic locomotion.

A. Bipedal Robot Model

Given the mass and inertia of the links and motors of AMBER2, the Euler-Lagrange equations yield the equations of motion, which can be converted to an affine control system of the form given in (3) (see [5], [17]). The discrete dynamics of AMBER2 describes the change of the states, i.e. the angles and angular velocities, after the non-stance foot impacts the ground. The end result is a hybrid system model of this system; additional details can be found in [1] and [10].

B. Output Design

To construct control Lyapunov functions, we will utilize the framework of human-inspired control [1], [15]. In particular, the end results of this method are relative 1 and 2 degree outputs of the form (see [2], [17] for additional details):

$$y_1(q, \dot{q}, v_{hip}) = y_{a,1}(q, \dot{q}) - v_{hip}, \tag{28}$$

$$y_2(q, \alpha) = y_{a,2}(q) - y_{d,2}(\tau(q), \alpha), \tag{29}$$

where $y_{a,1}(q, \dot{q})$ and $y_{a,2}(q)$ are the "actual" outputs, and $y_{d,2}(\tau(q), \alpha)$ is the "desired" output. It is important to note that the parameters α of $y_{d,2}$ are typically chosen through nonlinear optimization methods to yield *hybrid zero dynamics* and, thereby, guarantee a stable walking gait [10]. In this paper, we will instead choose α to be parameters obtained directly from human data [2] and utilize control barrier functions to achieve robotic walking.

C. Control Lyapunov Functions and Quadratic Programs

With the goal of driving $y_1 \to 0$ and $y_2 \to 0$, utilizing the methods from [3] and [5], the end result is a QP that unifies control objectives (CLFs), torque bounds, friction and Zero Moment Point (ZMP) constraints:

$$\begin{aligned}
\underset{(\delta, \bar{u}) \in \mathbb{R}^{11}}{\text{argmin}} \quad & \delta^T p \delta + \bar{u}^T \bar{A}^T \bar{A} \bar{u} + 2 L_f^T \bar{A} \bar{u} && (30)\\
\text{s.t.} \quad & \psi_{i,0}(q, \dot{q}) + \psi_{i,1}^T(q, \dot{q})(\bar{A}\bar{u} + L_f) \leq \delta_i, && \text{(CLF)}\\
& \dot{J}\dot{q} + JD(q)^{-1}(\bar{B}\bar{u} - H(q, \dot{q})) = 0, && \text{(Constrained Dynamics)}\\
& u \leq u_{max}, && \text{(Max Torque)}\\
& -u \leq u_{max}, && \text{(Min Torque)}\\
& -l_h F^{fz} < F^{my} < l_t F^{fz}, && \text{(ZMP)}\\
& |F^{fx}| < \mu_k F^{fz}, && \text{(Friction)}
\end{aligned}$$

where $i = \{1, 2, 3\}$, $p = \text{diag}(p_1, p_2, p_3)$ chosen to penalize relaxations δ_i of the CLF constraints. In addition, $u_{max} \in \mathbb{R}^6$ are maximum allowed torques, μ_k is the coefficient of static friction between AMBER2 and the ground, l_t is the length of the toe, l_h is the length of the heel, and $\bar{A}$, $\bar{u}$ and ψ are given as in [5].

IV. CBFs and Robotic Locomotion

In this section, we utilize the main formal results of this paper presented in Sect. II to construct control barrier functions in the case of robotic locomotion. In particular, each of the CLFs utilized in (30), (CLF1) - (CLF3), correspond to physical behavior on the robot: regulation of hip velocity, regulation of the non-stance foot height, the stance knee angle, and the forward movement of the non-stance leg (through the "non-stance slope" output [17]). Corresponding to each of these physical phenomena, we will construct a control barrier function that will ensure that the behavior of the robot satisfies physical constraints.

A. Control Barrier Functions

We will now construct control barrier functions enforcing physical constraints. Note that these constructions will utilize the fact that there is a natural correspondence between these constraints and a set $\mathcal{C}$ satisfying (4)-(6) where, in this case, the function $h(x)$ will correspond to elements the human-inspired outputs (28) and (29). As a result, the assumption of Theorem 4 are valid. In particular, we formulation (CBF1)-(CBF5) as (2) for every physical constraint. The following gives an overview of the specific CBFs considered, and the motivation for their constructions.

Hip velocity (CBF1): Since (CLF1), which corresponds to modulation of hip velocity, has relaxation parameter, the hip velocity will not converge to the desired velocity. Therefore, we should guarantee the velocity does not go below a lower bound, i.e., this CBF keeps the robot moving forward.

Non-stance foot height boundary (CBF2)-(CBF3): In order to ensure that the foot of the robot does not scuff the ground during a step, we must constraint the height of the foot to lie in a feasible region. Since the height of the foot is a function of the configuration variables of the system, the end result will be a function $h(x)$ of relative degree 2, so it is necessary to apply the methods introduced in Sect. II.

Stance knee angle (CBF4): In order to ensure that the stance knee does not hyper-extend, we introduce a physical constraint that the knee angle must be greater than 0.

Non-stance slope (CBF5): Since the CLF conditions (CLF1) and (CLF2) in (30) are relaxed, the corresponding outputs (in this case, the non-stance slope) will not track the desired values. As a result, a CBF constraint related to the non-stance slope can be constructed to ensure that, even in the presence of imperfect tracking, the non-stance foot will continue to move forward and thus the robot will complete a step. In particular, the non-stance slope will be constrained to lie under the desired non-stance slope: $y_H(\tau(q), \alpha_{nsl})$.

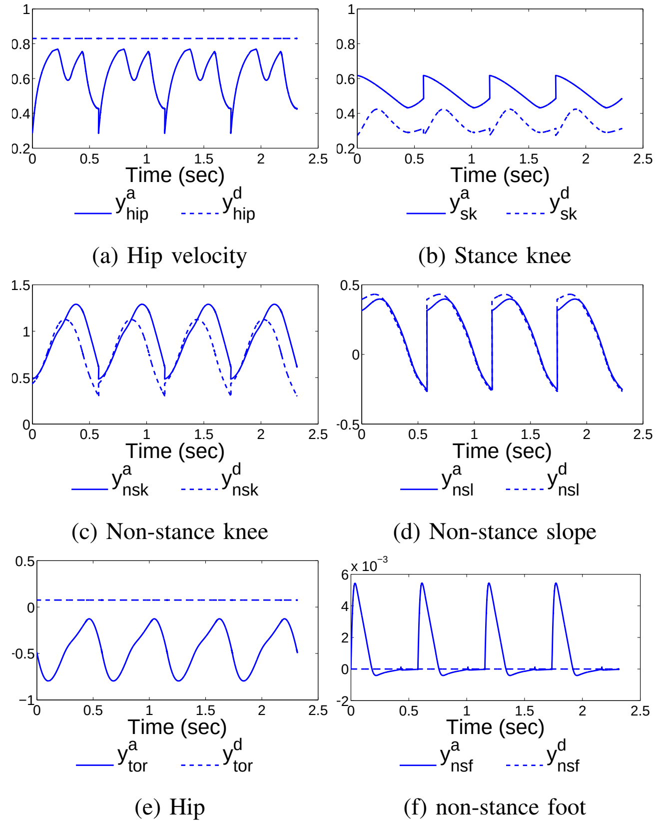
B. Results and Discussion

Adding the control barrier functions constructed in Sect. IV-A, as represented by constraints (CBF1)-(CBF5), to the control law presented in (30), the end result is a QP based controller that yields stable robotic walking. In particular, we use parameters v_{hip} and α of the human-inspired outputs (28) and (29) obtained by directly fitting the desired outputs to human data (see [1], [2], [17] for a more complete discussion). This is in contrast to the methods presented in [2], since we do not perform an *a priori* optimization to obtained parameters that guarantee (partial) hybrid zero dynamics [10]. If the control law (30) obtained from these outputs is simulated directly with the parameters v_{hip} and α obtained by fitting human data, the robot would stumble and fall. Yet, through the addition of the control barrier functions (CBF1)-(CBF5), the biped displays a stable walking gait (with the proper choice of parameters of the barrier functions); this points to the importance of enforcing physical constraints in the synthesis of robotic walking gaits.

In particular, to obtain a stable walking gait on the model of AMBER2, the parameters were chosen. The initial condition for the gait, $(q_0, \dot{q}_0) := (\theta(\alpha), \dot{\theta}(\alpha))$, was obtained by solving the inverse kinematics problem as outlined in [2]. The simulation result is illustrated in Fig. 1 to Fig. 4; importantly, the actual outputs do not converge to the desired outputs by design, i.e., the use of control barrier functions prevent exact convergence since they enforce physical constraints that dominate the control law. Yet a stable walking gait is still achieved, as evidenced by calculation of the eigenvalues of the Poincaré map—the maximum eigenvalues is 0.4432 (and hence smaller than 1) indicating stability.

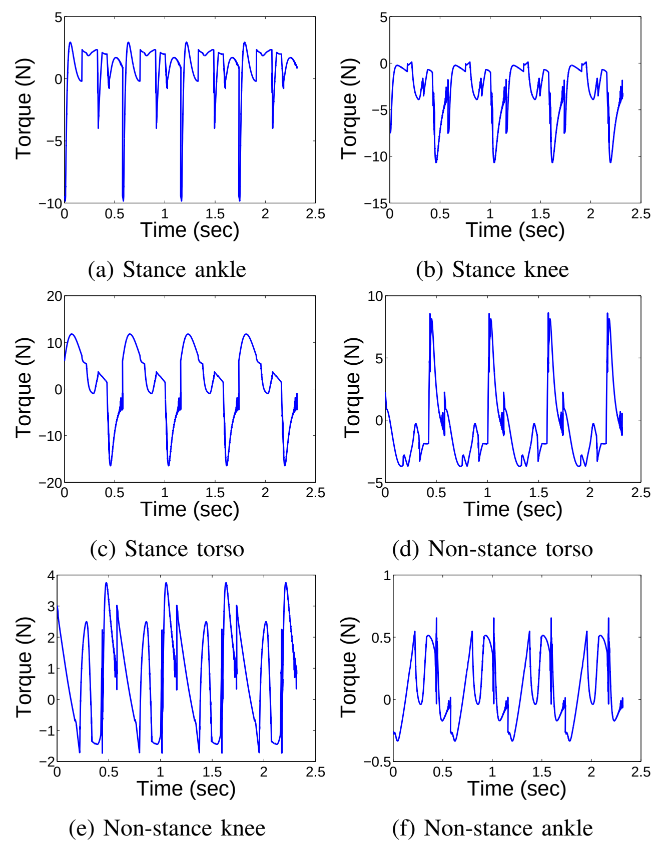
V. Conclusion

This paper presented a novel method for constructing control barrier functions through a backstepping inspired approach. In particular, we began by introducing a type of control barrier function that gives the maximum control authority (by allowing B to grow away from the boundary of the set $\mathcal{C}$); this allowed for the unification of safety constraints and control objectives through CLF-CBF based QPs. Yet the existence of control barrier functions of this form are not guaranteed to exist, i.e., there may not be control inputs that satisfied the required derivative conditions on the CBF. This motivated the main result of this paper: formal



(a) Hip velocity (b) Stance knee (c) Non-stance knee (d) Non-stance slope (e) Hip (f) non-stance foot

Fig. 1: Desired (dotted lines) and actual (solid lines) outputs during a stable periodic walking gait.



(a) Stance ankle (b) Stance knee (c) Stance torso (d) Non-stance torso (e) Non-stance knee (f) Non-stance ankle

Fig. 2: Torques on each joint during stable periodic walking.

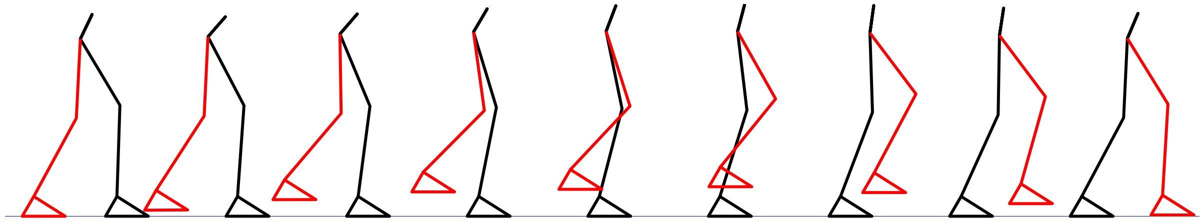
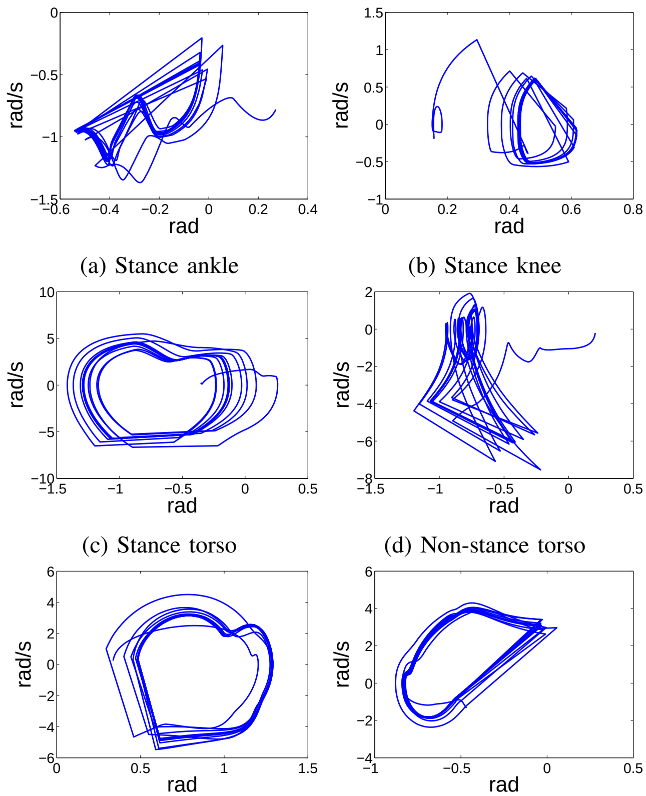


Fig. 4: Gait tiles for one step of a stable walking gait.



(a) Stance ankle (b) Stance knee (c) Stance torso (d) Non-stance torso (e) Non-stance knee (f) Non-stance ankle

Fig. 3: Phase portraits for each joint over 20 steps when started from an initial condition away from the fixed point; convergence to a stable periodic orbit can be seen.

guarantees on the existence of CBFs under assumptions on the relative degree of the function defining $\mathcal{C}$. To demonstrate the usefulness of these results, they were applied to bipedal robotic walking. Physical constraints that the robot must satisfy while locomoting were encoded as CBFs and combined with control objectives and torque/force constraints through a single QP based control law. The end result was stable walking in simulation.

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